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# Pairwise epistemic conditions for Nash equilibrium ☆

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ABSTRACT

We introduce a framework for modeling pairwise interactive beliefs and provide an epistemic foundation for Nash equilibrium in terms of pairwise epistemic conditions locally imposed on only some pairs of players. Our main result considerably weakens not only the standard sufficient conditions by Aumann and Brandenburger (1995), but also the subsequent generalization by Barelli (2009). Surprisingly, our conditions do not require nor imply mutual belief in rationality.



## 1. Introduction

In their seminal paper, Aumann and Brandenburger (1995) provided epistemic conditions for Nash equilibrium. Accordingly, if there exists a common prior, then mutual belief in rationality and payoffs as well as common belief in each player's conjecture about the opponents' strategies imply Nash equilibrium in normal form games with more than two players. As they pointed out, in their epistemic conditions *common knowledge*[1] *enters the picture in an unexpected way*; in fact, they stressed that *what is needed is common knowledge of the players' conjectures* and *not of the players' rationality* (Aumann and Brandenburger, 1995, p. 1163). Their result challenged the widespread view that common belief in rationality was essential for Nash equilibrium. Subsequently, Polak (1999) showed that in complete information games, Aumann and Brandenburger's conditions actually do imply common belief in rationality. In a sense, his result thus restored some of the initial confidence in the importance of common belief in rationality for Nash equilibrium. More recently, Barelli (2009) generalized Aumann and Brandenburger's result by substituting the common prior assumption with the weaker action-consistency property, and common belief in conjectures with a weaker condition stating that conjectures are constant in the support of

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[1] Aumann and Brandenburger (1995) actually use the term *knowledge* for probability-1 belief.

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the action-consistent distribution. Thus, he provided sufficient epistemic conditions for Nash equilibrium without requiring common belief in rationality, even in complete information games.

Here, we further generalize Aumann and Brandenburger's seminal result by introducing even weaker epistemic conditions for Nash equilibrium than those by Barelli (2009). Our results are based on introducing pairwise epistemic conditions imposed *only on some pairs of players*, contrary to the existing foundations of Aumann and Brandenburger (1995) and Barelli (2009) which correspond to pairwise epistemic conditions imposed *on all pairs of players*. Our general contribution consists of providing a general framework for modeling pairwise interactive beliefs of connected agents in a graph. Such a graph can be interpreted either as an auxiliary tool used to merely weaken the customary *global* epistemic conditions to *local* ones, or as a network representing physical connections between players. In the later case, our framework opens up the possibility to connect epistemic game theory with the theory of social networks, thus enabling a link between two independently developed streams of literature.

Our specific contribution consists of Theorem 1, in which we simultaneously replace (i) mutual belief in rationality with pairwise mutual belief in rationality, (ii) mutual belief in payoffs with pairwise mutual belief in payoffs, (iii) action-consistency with pairwise action-consistency, and (iv) constant conjectures in the support of the action-consistent distribution with pairwise constant conjectures in the support of the pairwise action-consistent distributions respectively, only for connected pairs of players. This difference is particularly important for large games – e.g. economies with many agents – where global epistemic conditions, such as requiring that every single player is certain that every other player is rational, can be rather demanding. In this respect, our assumptions are more plausible, as they impose pairwise conditions on relatively few pairs of players.

As a direct consequence of our main result, in Corollary 1, we also show that if a common prior exists, then pairwise mutual belief in rationality, pairwise mutual belief in payoffs and pairwise common belief in conjectures already suffice for a Nash equilibrium. The latter generalizes Aumann and Brandenburger (1995) in an orthogonal way compared to Barelli (2009). The following figure illustrates the relationship of our results to Aumann and Brandenburger (1995) and Barelli (2009).

| | | |
|---|---|---|
| Theorem 1 | ⇒ | Barelli (2009) |
| ⇓ | | ⇓ |
| Corollary 1 | ⇒ | Aumann and Brandenburger (1995) |

Apart from introducing a new framework and from providing a more general foundation for Nash equilibrium, we also contribute to the debate about the connection between common belief in rationality and Nash equilibrium. Indeed, since our conditions are weaker than Barelli's, they do not entail common belief in rationality even in complete information games. Surprisingly however, our conditions do not even require nor imply mutual belief in rationality. Thus, we reinforce Aumann and Brandenburger's intuition about common belief in rationality *not* being essential for Nash equilibrium, by showing that even mutual belief in rationality is not a crucial component. Moreover, as our corollary indicates, the absence of common belief in rationality from the epistemic conditions for Nash equilibrium should not be necessarily linked with the lack of a common prior, but instead it could be attributed to the fact that epistemic restrictions can be local, rather than global as it has been assumed in the literature so far.

## 2. Preliminaries

### *2.1. Normal form games*

Let $(I, (A_i)_{i\in I}, (g_i)_{i\in I})$ be a game in normal form, where $I = \{1, \dots, n\}$ denotes the finite set of players with typical element $i$, and $A_i$ denotes the finite set of strategies, also called actions, with typical element $a_i$ for every player $i \in I$. Moreover, define $A := \times_{i\in I} A_i$ with typical element $a := (a_1, \dots, a_n)$ and $A_{-i} := \times_{j\in I\setminus\{i\}} A_j$ with typical element $a_{-i} := (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$. The function $g_i : A_i \times A_{-i} \to \mathbb{R}$ denotes player $i$'s payoff function.

A probability measure $\phi_i \in \Delta(A_{-i})$ on the set of the opponents' action combinations is called a conjecture of $i$, with $\phi_i(a_{-i})$ signifying the probability that $i$ attributes to the opponents playing $a_{-i}$. Slightly abusing notation, let $\phi_i(a_j) := \text{marg}_{A_j} \phi_i(a_j)$ denote the probability that $i$ assigns to $j$ playing $a_j$. Note that it is standard to admit correlated beliefs, i.e. $\phi_i$ is not necessarily a product measure, hence the probability $\phi_i(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ can differ from the product $\phi_i(a_1) \cdots \phi_i(a_{i-1})\phi_i(a_{i+1}) \cdots \phi_i(a_n)$ of the marginal probabilities.[2] We say that an action $a_i$ is a best response to $\phi_i$, and write $a_i \in BR_i(\phi_i)$, whenever

$$\sum_{a_{-i}\in A_{-i}} \phi_i(a_{-i}) g_i(a_i, a_{-i}) \geqslant \sum_{a_{-i}\in A_{-i}} \phi_i(a_{-i}) g_i\left(a'_i, a_{-i}\right)$$

for all $a'_i \in A_i$.

[2] Intuitively, a player's belief on his opponents' choices can be correlated, even though players choose independently from each other.