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Continuous Optimization

A heuristic algorithm for constrained multi-source Weber problem – The variational inequality approach

Jian-Lin Jiang [a], Xiao-Ming Yuan [b,*,1]

[a] Department of Mathematics, College of Science, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China
[b] Department of Management Science, Antai College of Economics and Management, Shanghai Jiao Tong University, Shanghai 200052, China

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Abstract

For solving the well-known multi-source Weber problem (MWP), each iteration of the heuristic alternate location–allocation algorithm consists of a location phase and an allocation phase. The task of the location phase is to solve finitely many single-source Weber problems (SWP), which are reduced by the heuristic of nearest center reclassification for the customers in the previous allocation phase. This paper considers the more general and practical case – the MWP with constraints (CMWP). In particular, a variational inequality approach is contributed to solving the involved constrained SWP (CSWP), and thus a new heuristic algorithm for CMWP is presented. The involved CSWP in the location phases are reformulated into some linear variational inequalities, whose special structures lead to a new projection–contraction (PC) method. Global convergence of the PC method is proved under mild assumptions. The new heuristic algorithm using the PC method in the location phases approaches to the heuristic solution of CMWP efficiently, which is verified by the preliminary numerical results reported in this paper.

Keywords: Location–allocation; Constrained multi-source Weber problem; Variational inequalities; Projection–contraction

1. Introduction

The classical multi-source Weber problem (MWP) is to find the locations of m new facilities in order to minimize the sum of the transportation costs between these facilities and the n customers whose locations are known. We assume that the involved transportation costs are proportional to the corresponding distances. More specifically, the mathematical model of MWP is as follows:

* Corresponding author. Tel.: +86 21 52301397; fax: +86 21 62932982.
E-mail address: xmyuan@sjtu.edu.cn (X.-M. Yuan).
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$$\text{MWP}: \quad \min \sum_{j=1}^{n} \sum_{i=1}^{m} w_{ij} \|x_i - a_j\|$$
$$\sum_{i=1}^{m} w_{ij} = s_j, \quad j = 1, 2, \ldots, n,$$
$$x_i \in R^2, \quad i = 1, 2, \ldots, m,$$

where

(1) $a_j \in R^2$ is the location of the jth customer, $j = 1, 2, \ldots, n$;
(2) $x_i \in R^2$ is the location of the ith facility to be determined, $i = 1, 2, \ldots, m$;
(3) $s_j \geqslant 0$ is the given demand required by the jth customer;
(4) $w_{ij} \geqslant 0$ denotes the unknown allocation from the ith facility to the jth customer;
(5) $\|\cdot\|$ is the Euclidean distance.

Due to the wide applications of MWP in operations research, marketing, urban planing, etc., see, e.g, [6,10,17], many authors are devoted to both promoting its theoretical development and presenting effective numerical algorithms. To mention a few, the branch-and-bound algorithm presented in [19] obtains the solution by reducing MWP with $m = 2$ to $O(n^2)$ pairs of Weber problems; while the algorithm in [22] contributes a linear programming approach to solving MWP by enumerating all feasible elements of a solution (non-overlapping convex hulls). Among existing effective numerical algorithms for solving MWP is the heuristic alternate location–allocation algorithm presented originally in [5], whose attractive characteristic is that each iteration consists of a location phase and an allocation phase. Let $\mathcal{N} = \{1, 2, \ldots, n\}$ and $A = \{a_j : j \in \mathcal{N}\}$ denote the set of locations of all the customers and $\{A_1^k, A_2^k, \ldots, A_m^k\}$ with $\cup_{i=1}^{m} A_i^k = A$ and $A_i^k \cap A_j^k = \emptyset$ (for $i \neq j$) denote the disjoint partition of A at the kth iteration. At the $(k+1)$th iteration, the location phase finds the candidates of locations of facilities by solving m single-source Weber problems (SWP) denoted by

$$\text{SWP}: x_i^{k+1} = \operatorname*{argmin}_{x \in R^2} \left\{ C_i(x) := \sum_{\{j \in \mathcal{N} : a_j \in A_i^k\}} s_j \|x - a_j\| \right\}, \quad i = 1, 2, \ldots, m. \tag{1.1}$$

Then the allocation phase involves an allocation or a partition, which depends on the x_i^{k+1} $(i = 1, 2, \ldots, m)$ generated by solving (1.1). More specifically, if $\forall i \in \{1, 2, \ldots, m\}$, x_i^{k+1} is the nearest facility among all facilities for each customer in A_i^k, then x_i^{k+1} $(i = 1, 2, \ldots, m)$ are the desirable locations of facilities. Therefore, it is reasonable to allocate the customers in A_i^k from the facility x_i^{k+1} in order to minimize the total sum of transportation costs. Otherwise, the set of customers should be partitioned newly according to the heuristic of nearest center reclassification (NCR), i.e., the new partition $\{A_1^{k+1}, A_2^{k+1}, \ldots, A_m^{k+1}\}$ is generated so that x_i^{k+1} is the nearest facility for each customer in A_i^{k+1}. Note that with the presupposition that each facility to be determined is capable of providing sufficient services for the targeted customers, the heuristic characteristic shared by all Cooper-type algorithms is that finally each custom (a_j) is served only by one of the facilities (the nearest one). This observation explains the disappearance of w_{ij} in (1.1).

Then the central task of the location–allocation algorithm is to solve the involved SWP (1.1) in the location phase, and so it is meaningful to investigate effective numerical algorithms to solve (1.1). The first attractive contribution to this aspect denoted by Cooper's algorithm was due to [5], in which the author used the Weiszfeld procedure [25] to solve the involved SWP. Recently, the so-called Newton-Bracketing (NB) method for convex minimization was utilized to solve the involved SWP and thus Cooper-NB algorithm was developed in [16]. Due that the gradients of $\|x - a_j\|$ $(j = 1, 2, \ldots, n)$ are used in the iteration, both the Weiszfeld procedure and NB method share the common characteristic that their implementations may terminate unexpectedly when the current iterate happens to be identical with some location of the customers (which is unavoidable and uncontrollable), i.e. the singular case happens. How to improve the original Weiszfeld procedure and NB method in the singular case and make them computationally preferable become the main challenges in this study and still deserve more extensive investigations, see, e.g, [1,3,14,20,24]. For example, the effort proposed in [16] suggested to replace the gradient of $\|x - a_j\|$ with 0 whenever the singular case happens during the implementation.