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European Journal of Operational Research 193 (2009) 12–22

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

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Discrete Optimization

Subgradients of value functions in parametric dynamic programming ☆

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Received 7 November 2006; accepted 13 October 2007
Available online 24 October 2007

Abstract

We study in this paper the first-order behavior of value functions in parametric dynamic programming with linear constraints and nonconvex cost functions. By establishing an abstract result on the Fréchet subdifferential of value functions of parametric mathematical programming problems, some new formulas on the Fréchet subdifferential of value functions in parametric dynamic programming are obtained.

Keywords: Dynamic programming; Value functions; Fréchet normal cones; Fréchet subgradients; The Fréchet subdifferential

1. Introduction

A wide variety of problems in discrete optimal control can be posed in the following form.

Determine a control vector $u = (u_0, u_1, \ldots, u_{N-1}) \in U_0 \times U_1 \times \cdots \times U_{N-1}$ and a path $x = (x_0, x_1, \ldots, x_N) \in X_0 \times X_1 \times \cdots \times X_N$ which minimize the cost

$$\sum_{k=0}^{N-1} h_k(x_k, u_k, w_k) + h_N(x_N) \tag{1}$$

with state equation

$$x_{k+1} = A_k x_k + B_k u_k + T_k w_k \quad \text{for all } k = 0, 1, \ldots, N-1, \tag{2}$$

the constraints

$$u_k \in \Omega_k \subset X_k \quad \text{for all } k = 0, 1, \ldots, N-1 \tag{3}$$

and initial condition

$$x_0 = x \in X_0, \tag{4}$$

where

k indexes discrete time,
x_k is the state of the system and summarizes past information that is relevant for future optimization,

☆ This research was partially supported by a grant from National Science Council of Taiwan, ROC.

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0377-2217/$ - see front matter
doi:10.1016/j.ejor.2007.10.028

u_k is the control variable to be selected at time k with knowledge of the state x_k,
w_k is a random parameter (also called disturbance or noise),
N is the horizon or number times control is applied,
X_k is a finite-dimensional space of state variables at stage k,
U_k is a finite-dimensional space of control variables at stage k,
Ω_k is a nonempty set in U_k.
W_k is a finite-dimensional space of random parameters at stage k.

A classical example for the problems (1)–(4) is the *inventory control problem* where x_k plays a stock available at the beginning of the kth period; u_k plays a stock order at the beginning of the kth period and w_k is the demand during the kth period with given probability distribution; and the cost function has the form $\sum_{k=0}^{N-1} cu_k + H(x_k + u_k - w_k)$ together with state equation $x_{k+1} = x_k + u_k - w_k$ (see [3] for details). For more information and necessary conditions of problem (1)–(4), we refer the reader to [2,3,7,9–13,19].

Put $W = W_0 \times W_1 \times \cdots \times W_{N-1}$, $X = X_0 \times X_1 \times \cdots \times X_N$ and $U = U_0 \times U_1 \times \cdots \times U_{N-1}$. We denote by $V(w)$ the optimal value of the problem (1)–(4) corresponding to the parameter $w = (w_0, w_1, \ldots, w_{N-1}) \in W$. Thus $V : W \to \overline{R}$ is an extended real-valued function which is called the value function of the problems (1)–(4).

The study of first-order behavior of value functions is of importance in analysis and optimization. An example of this type is distance functions and its applications in optimal control problems with differential inclusions (e.g., [1,8,26]). There have been many papers dealing with differentiability properties and the Fréchet subdifferential of value functions in the literature (e.g., [6,16–18,21]). Under Lipschitzian conditions and the assumption that the solution set of perturbed problems is nonempty in a neighborhood of an unperturbed problem, Clarke [6, Theorem 6.52] established a useful formula for the generalized gradient of a value function. By considering a set of assumptions which involves a kind of coherence property, Penot [21] showed that the value functions are Fréchet differentiable. The results of Penot gave sufficient conditions under which the value functions are Fréchet differentiable rather than formulas computing their derivatives. In [16], Mordukhovich, Nam and Yen derived formulas for computing and estimating the so-called Fréchet subdifferential of value functions of parametric mathematical programming problems in Banach spaces without Lipschitzian assumptions.

Besides the study of first-order behavior of value functions in parametric mathematical programming, the study of first-order behavior of value functions in optimal control is also of interest of many researchers. We refer the reader to [20,22–25] for recent studies on sensitivity analysis of the optimal-value function in parametric optimal control. In particular, Seeger [24] obtained a formula for the approximate subdifferential of convex analysis of V to the case where h_k and Ω_k were assumed to be convex, and the problem can have no optimal paths. It is noted that if Ω_k and h_k are convex for all $k = 0, 1, \ldots, N$, then V becomes a convex function. In this case, we can compute the subdifferential of V via subgradients of convex functions. However, the situation will be more complicated if h_k and Ω_k are not convex because subgradients of convex functions fail to apply.

It is well recognized that the function V can fail to be smooth despite any degree of smoothness of h_k. The aim of this paper is to derive some new formula for computing the so-called Fréchet subdifferential of V via the tool of generalized differentiation. In order to obtain the result, we first establish a formula for computing and estimating the Fréchet subdifferential of the value functions for a special class of parametric mathematical programming problems (Theorem 2.1). Then we apply Theorem 2.1 to prove Theorem 1.1 which is the main result of this paper. Our proof of Theorem 2.1 closely follows the method of [16]. However, we dealt with the formula of basic normals to set intersections in the product of Asplund spaces and establish a formula for computing the normal cone of constraints sets.

Let us recall some notions related to generalized differentiation. The notions and results of generalized differentiation can be found in [4,5,14,15]. Let $\varphi : Z \to \overline{R}$ be a extended real-valued function on a finite-dimensional space Z and $\bar{x} \in Z$ be such that $\varphi(\bar{x})$ is finite. The set

$$\hat{\partial}\varphi(\bar{x}) := \left\{ x^* \in X \,\middle|\, \liminf_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geqslant 0 \right\} \tag{5}$$

is called the Fréchet subdifferential of φ at $\bar{x}$. A vector $x^* \in \widehat{\partial}\varphi(\bar{x})$ is called a Fréchet subgradient of φ at $\bar{x}$. It is known that the Fréchet subdifferential reduces to the classical Fréchet derivative for differentiable functions and to subdifferential of convex analysis for convex functions. The set $\hat{\partial}^+\varphi(\bar{x}) := -\hat{\partial}(-\varphi)(\bar{x})$ is called the *upper subdifferential* of φ at $\bar{x}$.

Let Ω be a nonempty set in Z. Given $\bar{z} \in \Omega$ and $\epsilon \geqslant 0$, define the *set of ϵ-normal* by

$$\widehat{N}_\epsilon(\bar{z}; \Omega) := \left\{ z^* \in Z^* \,\middle|\, \limsup_{z \to \bar{z}} \frac{\langle z^*, z - \bar{z} \rangle}{\|z - \bar{z}\|} \leqslant \epsilon \right\}. \tag{6}$$

When $\epsilon = 0$, the set $\widehat{N}(\bar{z}; \Omega) := \widehat{N}_0(\bar{z}; \Omega)$ is called the Fréchet normal cone to Ω at $\bar{z}$. It is also well known that if $\delta(z, \Omega)$ is the indicator function of Ω, i.e., $\delta(z, \Omega) := 0$ if $z \in \Omega$ and $\delta(z, \Omega) := \infty$ otherwise, then $\widehat{N}(\bar{z}; \Omega) = \widehat{\partial}\delta(\bar{z}; \Omega)$. A vector $z^* \in Z^*$ (the dual space of Z) is called a *limiting normal* to Ω at $\bar{z}$ if there exist sequences $\epsilon_k \to 0^+$, $z_k \to \bar{z}$, and $z_k^* \to z^*$ such that