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Localisation of continuity to bounded sets for nonmetrisable vector topologies and its applications to economic equilibrium theory‡

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ABSTRACT

We present two 'localisation' techniques which facilitate verifications of the topological properties of sets, functions and correspondences needed in, e.g., economic equilibrium analysis with infinite-dimensional commodity spaces. The first uses the Krein-Smulian theorem, which shows that weak* upper semicontinuity of a concave function on a dual Banach space is equivalent to bounded weak* u.s. continuity. The second is based on the continuity of lattice operations: for a nondecreasing function on a topological vector lattice, we show that lower semicontinuity on a set bounded from below is equivalent to l.s. continuity on bounded subsets. In the case of L^∞, we use convergence in measure to establish that *sequential* semicontinuity, lower for the Mackey topology or upper for the weak* one, is equivalent to semicontinuity. This greatly simplifies some arguments; e.g., the Mackey continuity of a concave, nondecreasing integral functional on L^∞_+ becomes an immediate consequence of Lebesgue's Dominated Convergence Theorem. Other uses in mathematical economics are also discussed.

1. INTRODUCTION

The use of nonmetrisable topologies, especially the weak* and the Mackey topologies, is essential in much of modern mathematical economics, e.g., in competitive equilibrium theory – and more generally whenever optimisation over an infinite-dimensional dual Banach space L is involved. These topologies can be metrisable on bounded regions of L; and in such cases they are much easier to work with if their use can be restricted to bounded regions – and thus

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to *ordinary sequences*, as distinct from uncountable nets (a.k.a. generalised sequences). This is because the sequential or, equivalently, 'bounded' variants of continuity and closedness conditions are much easier to verify directly than are their 'full' variants. And when it suffices to verify continuity along sequences of certain special forms, this can also help clarify economic interpretations of continuity conditions.

On some occasions it *is* clear without much ado that the apparently weaker conditions will suffice for the results being sought. As an example in mathematical economics, to establish equilibrium existence and specific representations of price systems, consumer preferences and production sets are assumed to be continuous or closed for appropriate topologies. Although this is not always pointed out in the literature, some arguments using nonmetrisable topologies actually involve only the bounded subsets of L: examples include Bewley's study of the commodity space L^∞ [1, Theorems 1 to 3]. But some other results, though closely related, do rely on the use of unbounded sets; and this raises the question of whether the restricted (or sequential) continuity or closedness does actually imply the property in full.[1]

Our objective is, then, to identify some cases in which continuity properties do localise to bounded regions of the space (i.e., follow from the corresponding properties of a function's restrictions to bounded sets). Such results are particularly useful when the topologies in question, despite being nonmetrisable globally, *are* metrisable on the *bounded* subsets. One can then completely avoid dealing with general, uncountable nets – and still establish unqualified continuity.

We present two quite different localisation techniques for establishing this. One is essentially the Krein-Smulian theorem, which applies to convex subsets of a dual Banach space with the weak* topology (Proposition 1). The other method is based on a monotonicity argument, and it applies to certain subsets of a topological vector lattice, also nonconvex ones (Proposition 3). Both methods apply to $L = L^\infty$ (Examples 2 and 4). With this space it is also very useful that the Mackey topology is equivalent, on bounded subsets, to the topology of convergence in measure: e.g., it follows that the Mackey continuity of a real-valued, nondecreasing and concave function on L^∞_+ is equivalent to continuity along bounded sequences convergent in measure. This greatly simplifies some arguments. For example, the Mackey continuity of concave integral functionals becomes an immediate consequence of Lebesgue's Dominated Convergence Theorem (Example 5) – by contrast with the lengthy and complicated proof in [1, pp. 535–539]. Note that the reduction to ordinary sequences is essential, since Lebesgue's theorem does *not* apply to uncountable nets.

As has already been mentioned, localisation to bounded regions is closely connected to the sufficiency of sequences, since the nonmetrisable topologies in question can be metrisable on bounded sets. In such cases localisation means

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[1]For example, Florenzano's [3] extension of the Debreu-Gale-Nikaido Lemma requires the closedness of the *whole* production cone, and not just of its bounded parts.