J. Math. Anal. Appl. 351 (2009) 675–681

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Topological vector spaces admissible in economic equilibrium theory

Hans Keiding

University of Copenhagen, Department of Economics, Studiestraede 6, DK-1455 Copenhagen K., Denmark

ARTICLE INFO

Article history:
Received 5 August 2008
Available online 6 November 2008
Submitted by J.A. Filar

Keywords:
Topological vector spaces
Equilibrium theory
Kakutani space

ABSTRACT

In models of economic equilibrium in markets with infinitely many commodities, the commodity space is an ordered topological vector space endowed with additional structure. In the present paper, we consider ordered topological vector spaces which are admissible (for equilibrium analysis) in the sense that every economy which is reasonably well behaved posesses an equilibrium. It turns out that this condition may be characterized in terms of topology and order. This characterization implies that the commodity space has the structure of a Kakutani space.

1. Introduction

The economic theory of markets with an infinite number of commodities has been the subject of a considerable literature, starting with the seminal work of Bewley [8] and followed by works of Aliprantis and Brown [1], Florenzano [10], Mas-Colell [13], Yannelis and Zame [17], Araujo and Monteiro [6], and more recently Aliprantis, Monteiro and Tourky [5], Aliprantis, Florenzano and Tourky [4]. Many of the results obtained have appeared also in monographs (Aliprantis, Brown and Burkinshaw [2], Florenzano [11]).

The generalization of the standard model of competitive equilibrium to markets with infinitely many commodities is not a straightforward matter. Indeed, replacing Euclidean space with an arbitrary ordered topological vector spaces meets with several problems; closed and order bounded sets are not necessarily compact, and the equilibrium prices are not necessarily continuous. These difficulties—and several others—have been pointed out in the literature, and results on existence of competitive equilibria have been established, but only for spaces satisfying several conditions of well-behavedness, some of which using assumptions on agents' characteristics with no counterpart in models with a finite number of commodities (such as the assumption of properness of preferences, cf. Mas-Colell [13]).

These results indicate that certain properties of ordered topological vector spaces are crucial, such as compactness of order intervals in the weak topology, and existence of interior points in the positive cone. In the present paper we make this precise, investigating commodity spaces which can be used in a formal theory of economic equilibrium with infinitely many commodities ("admissible" commodity spaces), in the sense that for every well-behaved economy there is at least one equilibrium.

These properties, which may be considered as necessary conditions on the underlying topological vector spaces for being eligible as commodity spaces in models of economic equilibrium, relate to compactness of order intervals and existence of order units, but they do *not* involve lattice properties of the order. Indeed, early contributions to existence theory, such as Aliprantis and Brown [1], Yannelis [16], did not use a lattice structure, and it became necessary only when considering core equivalence theorems, as shown by Aliprantis and Burkinshaw [3].

The paper is organized as follows: In Section 2, we present the necessary definitions, and in Section 3, we state and prove the main results of the paper, deriving necessary conditions for admissibility, and in Section 4 it is shown that these

E-mail address: hans.keiding@econ.ku.dk.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.11.004

necessary conditions are also sufficient. Section 5 gives a characterization of the admissible commodity spaces for vector lattices, and finally Section 6 contains some concluding comments on the approach and the results.

2. Definitions

In this section, we introduce the basic concepts of the paper, which are the notion of a commodity spaces and the notion of an admissible commodity space.

Definition 1. A commodity space is an ordered Hausdorff topological vector space V, where the order relation $\leqslant$ is defined by a convex closed cone V_+ such that

(i) $V_+ - V_+ = V$,
(ii) if $v_1, v_2 \in V$ and $nv_1 \leqslant v_2$ for all $n \in \mathbb{N}$, then $v_1 \leqslant 0$.

We notice that a commodity space need not (at least at this stage) be a lattice nor a Riesz space; we assume only that the every element $v \in V$ has a representation $v = v_1 - v_2$, $v_1, v_2 \in V_+$, not that the representation is unique.

Let $(V_+)^*$ be the polar of the cone V_+, that is the set of linear forms p on V such that $p(v) \geqslant 0$ for all $v \in V_+$. The set $V^+ = (V_+)^* - (V_+)^*$ is called the order dual of V.

In order to introduce our notion of an admissible commodity space, we define economies over a commodity space and equilibria for these economies.

Definition 2. Let V be a commodity space. An economy over V is a finite array $\mathcal{E} = (P_i, \omega_i)_{i\in I}$ of consumers, where for each consumer $i \in I$, $P_i \subset V_+$, V_+ is a (preference) relation on V_+, and $\omega_i \in V$ an initial endowment of commodities.

An economy $\mathcal{E}$ is well behaved if

(i) the vector ω_i belongs to V_+ for each $i \in I$,
(ii) for all $i \in I$, P_i satisfies
  (a) $P_i(x_i) = \{x_i' \in V_+ \mid (x_i, x_i') \in P_i\}$ is convex, and $x_i \notin P_i(x_i)$, each $x_i \in V_+$,
  (b) for each $x_i' \in V_+$, the set $P^{-1}(x_i') \cap [0,\omega]$, where $P_i^{-1}(x_i') = \{x_i \in V_+ \mid (x_i, x_i') \in P_i\}$, is $\sigma(V, V^+)$-open in $[0,\omega]$,
(iii) there is a consumer $i^0 \in I$ such that for all $x_{i^0} \in V_+$, $P_{i^0}(x_{i^0}) + V_+ \subset P_{i^0}(x_{i^0})$.

The notion of an economy is fairly standard, but the additional conditions of well-behavedness may require some comments. It is assumed that set of feasible consumptions of each consumer is V_+, ruling out the possibility of consumption sets contained in proper subspaces of V. We allow for non-ordered preferences of consumers since this is by now commonplace in equilibrium theory. Condition (i) is a weak survival condition on individual endowments; condition (ii)(b) states the standard properties of convexity and irreflexivity of preferences, and (ii)(a) is a continuity property; it is a rather weak assumption, as the openness of $P_i^{-1}(x_i')$ should hold only relative to the set of vectors effectively attainable by consumer i in $\mathcal{E}$, namely the vectors in the order interval $[0,\omega] = \{x_i \mid 0 \leqslant x_i \leqslant \omega\}$. Finally, condition (iii) says that some consumers have monotonic preferences.

Definition 3. Let V be a commodity space, $\mathcal{E}$ an economy over V. A quasi-equilibrium is a pair $(x, p) \in V_+^I \times (V_+)^*$, such that

(i) $\sum_{i\in I} x_i \leqslant \sum_{i\in I} \omega_i$ (aggregate feasibility),
(ii) for all $i \in I$, $p(x_i) \leqslant p(\omega_i)$, and $x_i' \in P(x_i)$ implies $p(x_i') \geqslant p(\omega_i)$ (individual optimality),
(iii) $p(\sum_{i\in I} \omega_i) > 0$, or there exists $q \in (V_+)^*$ such that for each $i \in I$, $q(x_i) \leqslant q(\omega_i)$ and $x_i' \in P_i(x_i) \cap \operatorname{Ker} p$ implies $q(x_i') \geqslant q(\omega_i)$.

The equilibrium conditions state that the allocation x can be carried out with the given resources, and that consumers get a bundle which is best given their budgets in the sense that any preferred bundle is at least as expensive. The condition (iii) is formulated in a way so as to avoid trivial quasi-equilibria where value of every bundle is 0.

Now we may define the central concept of this paper, that of an admissible commodity space.

Definition 4. Let V be a commodity space. V is admissible if every economy $\mathcal{E} = (P_i, \omega_i)_{i\in I}$ over V has a quasi-equilibrium.

A commodity space is admissible if it can be used in modelling a classical economic equilibrium theory in such a way that equilibria always exist. Needless to say, if further properties are added to the definition of an economy (as in Mas-Colell [13]), or the definition of an equilibrium is weakened (as in Aliprantis, Monteiro and Touky [4]), more commodity spaces are available. In the next section, we exhibit properties of commodity spaces which are admissible according to Definition 4.