Nonlinear Analysis 74 (2011) 5048–5056

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Lagrangean variables in infinite dimensional spaces for a dynamic economic equilibrium problem

Maria Bernadette Donato, Monica Milasi *

Department of Mathematics, University of Messina, Viale Ferdinando Stagno d'Alcontres, 31 - 98166 Messina, Italy

ARTICLE INFO

Article history:
Received 3 December 2010
Accepted 28 April 2011
Communicated by S. Carl

Keywords:
Competitive equilibrium
Evolutionary variational inequality
Lagrangean multipliers

ABSTRACT

This paper is focused on the study of a dynamic competitive equilibrium by using Lagrangean multipliers. This mathematical formulation allows us the improve the Walrasian model by considering the common possibility of an uncharged delayed payment in a given time (for example, by using a credit card). Firstly the economic equilibrium problem is reformulated as an evolutionary variational problem; then the Lagrangean theory in infinite dimensional spaces is applied. Thanks to the application of this theory we obtain the existence of Lagrangean multipliers, which allows us to give a computational procedure for the equilibrium solutions.

1. Introduction

The duality and Lagrangean theory provide interesting contributions, absolutely necessary for the better understanding and handling of an increasing number of equilibrium problems whether in the static case or in the evolutionary case: the traffic equilibrium problem, the financial equilibrium problem, the spatially distributed markets equilibrium problem, the oligopolistic market equilibrium problem, the mixed equilibrium problem, the pollution control problem, the vaccination problem, and the Walras equilibrium problem (see e.g. [1–10]). Each of the aforementioned problems can be transformed into a variational problem on a convex subset K of a suitable functional space, for which an impressive quantity of results in terms of existence, calculation of the solutions, stability and sensitivity analysis hold. This study is also undertaken by applying a suitable duality and Lagrangean theory in order to obtain necessary and/or sufficient optimality conditions. The Lagrangean theory is useful not only to provide a qualitative analysis of equilibria, but also to calculate the solutions.

In this paper, a dynamic competitive equilibrium for a Walrasian pure exchange economy has been considered. This problem was introduced by Walras in [11]. The equilibrium model is built through the maximization of a utility function from which a nonpositive excess demand function is derived. Still now the mathematical study of the Walras equilibrium represents an active and interesting research topic (see e.g. [12–15,8,16–18]). In this paper, we introduce a continuum model of a competitive equilibrium for a pure exchange economy in the finite interval $[0, T]$ and we give a dynamic formulation of the equilibrium conditions. In this evolutionary market the aim of each agent is not to maximize his utility at the fixed instant t, but to maximize it globally in the whole period of time. During the interval of time $[0, T]$ each agent trades his own commodities with the other agents taking into account his own budget set: the amount that each agent pays for acquiring the goods in the whole period $[0, T]$ is at most the amount that each agent receives, during the period $[0, T]$, as his initial endowment. Then, mathematically the equilibrium is reformulated in terms of a maximization problem of an integral utility function on a feasible set, in which the constraint is given in an integral form and in terms of a nonpositive excess demand function. First, we prove that the equilibrium is a solution to a suitable evolutionary quasi-variational inequality which is set in the Lebesgue space $L^2([0, T], \mathbb{R})$. Our main result in this paper is to obtain the Lagrangean variables relative to the

* Corresponding author. Tel.: +39 3495863507; fax: +39 090393502.
E-mail addresses: mmilasi@unime.it, monica.milasi@gmail.com (M. Milasi).

0362-546X/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2011.04.070

equilibrium, by applying the Lagrangean theory in infinite dimensional normed spaces. These variables allow us to have a complete description of the behaviour of the economic market during the evolution in time. We would like to emphasize that, since the budget constraint is expressed in an integral form, it is not possible to obtain the equivalence between the evolutionary variational problem and a pointwise variational problem in a finite dimensional space. So we cannot calculate equilibrium solutions by solving the pointwise variational inequality. Our study will also be useful to overcome the difficulty of computing equilibrium solutions. The paper is organized as follows: in Section 2, we introduce the evolution in time of a competitive equilibrium problem for a pure exchange economy. In Section 3, we prove that the evolutionary equilibrium is a solution to a suitable evolutionary quasi-variational inequality and, by using this variational formulation, we apply the Lagrangean theory to our problem. Finally, the equilibrium is characterized in terms of a quasi-variational inequality and in terms of the Lagrange multipliers. In Section 4, we conclude the paper with some remarks and, in order to support the theoretical results, we provide a numerical example. So this numerical example will provide evidence of the efficacy of our main result.

2. Dynamic competitive equilibrium problem

Let us consider a marketplace consisting of l different goods indexed by j and n agents indexed by a in the period of time $[0, T]$. In order to express the time-dependent equilibrium conditions by means of an evolutionary variational inequality, we choose as a functional setting the Lebesgue space $L^2([0, T], \mathbb{R}^l)$. Let $e_a^j(t)$ and $x_a^j(t)$ denote, respectively, the nonnegative endowment and consumption by agent a of the commodity j at the time t. We suppose that each agent a is at least endowed with a positive quantity of commodity (survivability assumption). Let $p^j(t)$ denote the nonnegative price associated to commodity j at time t. By grouping the introduced quantities in vectors one has that the total endowment vector e_a, the consumption vector x_a and the price vector p belong in $L^2([0, T], \mathbb{R}^l) = L$ and the market consumption matrix $x = \{x_a\}_{a=1,\dots,n}$ belongs to $L^2([0, T], \mathbb{R}^{n\times l})$. In this market, where a competitive behaviour prevails, the agents' preferences, relative to the total consumption x_a at time t, are expressed by the utility functions $u_a(t, x_a(t))$ defined on $[0, T] \times \mathbb{R}^l$. We suppose that, for all $a = 1, \dots, n$,

(A_1) $u_a(t, x_a(t))$ is measurable in t;
(A_2) $u_a(t, x_a)$ is concave with respect to x_a;
(A_3) for all $j = 1, \dots, l$, $\frac{\partial u_a(t,x_a)}{\partial x_a^j}$ exists and is measurable in t and continuous with respect to x_a;
(A_4) the following growth condition holds:

$$\| -\nabla u_a(t, x_a(t))\| \leq \mu_a(t)\|x_a\|_L + \nu_a(t) \quad \forall x_a \in L \text{ a.e. in } [0, T]$$

where $\mu_a \in L^\infty([0, T])$ and $\nu_a \in L^2([0, T])$ are nonnegative functions.

Since the utility function in an economy is usually concave, then we require the condition (A_2), so that the integral utility function $\mathcal{U}_a(x_a) = \int_0^T u_a(t, x_a(t))\, dt$ is concave. Thanks to conditions (A_1), (A_3), (A_4) we can characterize the dynamic problem in terms of a variational inequality problem involving the L^2-scalar product in $[0, T]$.

Let C_L be the ordering cone of L: $C_L = \{\alpha \in L : \alpha^j(t) \geq 0 \text{ a.e. in } [0, T]\, \forall j = 1, \dots, l\}$. Now let us give the following definition of an equilibrium for a dynamic competitive economy.

Definition 1. The pair $(\overline{p}, \overline{x}) \in P \times \prod_{a=1}^n M_a(\overline{p})$ is a dynamic competitive equilibrium if and only if

$$\text{for all } a = 1, \dots, n \quad \mathcal{U}_a(\overline{x}_a) = \max_{x_a \in M_a(\overline{p})} \int_0^T u_a(t, x_a(t))\, dt, \tag{1}$$

$$\text{for all } j = 1, \dots, l \quad \sum_{a=1}^n \left[\overline{x}_a^j(t) - e_a^j(t)\right] \leq 0 \quad \text{a.e. in } [0, T], \tag{2}$$

where $M_a(\overline{p}) = \{x_a \in L : x_a \in C_L, \langle \overline{p}, x_a - e_a\rangle_L \leq 0\}$, and $P = \left\{p \in L : p \in C_L, \frac{1}{T}\int_0^T \sum_{j=1}^l p^j(t)dt = 1\right\}$.

In the set $M_a(\overline{p})$ the inner product $\langle \overline{p}, x_a\rangle_L$ represents how much the agent spends in all the time $[0, T]$ to buy commodity x_a; while the inner product $\langle \overline{p}, e_a\rangle_L$ represents how much the agent earnes during the interval of time $[0, T]$, by selling his endowment e_a. This trading is made taking into account the current price $\overline{p}$. Hence, since there are not further profits, each consumer must satisfy the natural budgetary constraint: the selling cannot exceed the earning; this constraint is not required at the fixed time t, but globally in the whole time $[0, T]$.

Here we set the price as an average with respect to the time, that is $\frac{1}{T}\int_0^T \sum_{j=1}^l p^j(t)\, dt = 1$. This approach is inspired from the dynamic traffic equilibrium model with elastic demand given in [2] and by taking into account that the dynamic Walrasian price equilibrium problem can be considered as a particular network equilibrium problem (see e.g. [19]). The meaning of the average formulation on the constraint set P is that prices can float in the considered time frame. The prices might exceed 1 at a certain time, and later be less than 1 so that on average they are exactly 1. The latter condition is a realistic addition to Walrasian models since the prices are flexible, as happens in the real world.