Insurance: Mathematics and Economics 51 (2012) 191–203

Contents lists available at SciVerse ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Optimal time-consistent investment and reinsurance strategies for insurers under Heston's SV model[☆]

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ARTICLE INFO

Article history:
Received May 2011
Received in revised form
August 2011
Accepted 9 September 2011

MSC:
IM52
IE13
IB91

Keywords:
Investment and reinsurance strategy
Stochastic volatility
Time-consistency
Insurer
Mean–variance criterion

ABSTRACT

This paper considers the optimal time-consistent investment and reinsurance strategies for an insurer under Heston's stochastic volatility (SV) model. Such an SV model applied to insurers' portfolio problems has not yet been discussed as far as we know. The surplus process of the insurer is approximated by a Brownian motion with drift. The financial market consists of one risk-free asset and one risky asset whose price process satisfies Heston's SV model. Firstly, a general problem is formulated and a verification theorem is provided. Secondly, the closed-form expressions of the optimal strategies and the optimal value functions for the mean–variance problem without precommitment are derived under two cases: one is the investment–reinsurance case and the other is the investment-only case. Thirdly, economic implications and numerical sensitivity analysis are presented for our results. Finally, some interesting phenomena are found and discussed.

1. Introduction

Due to the facts that reinsurance is an effective way to spread risk and that investment is an increasingly important element in the insurance business, optimal investment and reinsurance problems for insurers have drawn great attention in recent years. For example, Browne (1995), Yang and Zhang (2005), Wang (2007) and Xu et al. (2008) studied the optimal investment strategies to maximize insurers' expected exponential utility from the terminal wealth under different market assumptions; Bai and Guo (2008), Luo et al. (2008), Luo (2009) and Azcue and Muler (2009) investigated the optimal investment and reinsurance strategies to minimize the ruin probability of insurers in different situations. In addition, some scholars have recently studied the optimal investment and reinsurance strategies for insurers under the mean-variance criterion proposed by Markowitz (1952), see among others, Bäuerle (2005), Delong and Gerrard (2007), Bai and Zhang (2008), Zeng et al. (2010) and Zeng and Li (2011).

However, two aspects are worthy to be further explored based on the above-mentioned literature. On the one hand, the price processes of risky assets in most of the above-mentioned literature are assumed to follow geometric Brownian motion. Hence, the volatilities of risky assets are constant or deterministic, which is contrary to the fact. Stochastic volatility (SV) has been recognized recently as an important feature for asset price models. There is much literature documenting SV in assets' returns, such as French et al. (1987) and Pagan and Schwert (1990). Meanwhile, SV can be seen as an explanation of many well-known empirical findings, such as the volatility smile, the volatility clustering, and the heavy-tailed nature of return distributions. Many scholars have studied the optimal investment and/or consumption problems under the expected utility criterion with SV models. For example, Zariphopoulou (1999), Fleming and Hernández-Hernández (2003), Chacko and Viceira (2005) and Liu (2007) considered the optimal investment and consumption problems under SV models and derived the closed-form expression of the optimal strategies and the optimal value functions in some situations using the HJB approach; Zariphopoulou (2001), Pham (2002), Kraft (2005) and Taksar and Zeng (2009) studied the optimal investment problems

☆ This research is supported by grants from the National Science Foundation for Distinguished Young Scholars (No. 70825002), the National Basic Research Program of China (973 Program, No. 2007CB814902), and "985 Project" of Sun Yat-sen University.

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doi:10.1016/j.insmatheco.2011.09.002

under SV models. Moreover, Korn and Kraft (2001) solved the optimal investment problems with stochastic interest rates and derived the optimal strategies to maximize the expected utility from the terminal wealth using a stochastic control method; Ferland and Watier (2010) considered a mean-variance investment problem with the Cox–Ingersoll–Ross (CIR) interest rate in a continuous-time framework and constructed a mean-variance efficient portfolio through the solution of backward stochastic differential equations. Li and Wu (2009) and Noh and Kim (2010) studied the optimal investment problems with an SV asset price process and a stochastic interest rate to maximize the expected utility from the terminal wealth.

On the other hand, the optimal strategies of the above-mentioned literature under the mean-variance criterion are not time-consistent in the sense that, if they are optimal at the initial time, they are also optimal in any remaining time interval. There are two reasons for this. First, the mean-variance criterion lacks the iterated expectation property; hence, continuous-time and multiperiod mean-variance problems are time-inconsistent in the sense that the Bellman's principle of optimality does not hold. Second, the optimal strategies are derived under the assumption that the investors precommit themselves not to deviate from the strategies chosen at the initial time. However, in many situations time-consistency of strategies is a basic requirement for rational decision-makers. Strotz (1956) proposed that time-inconsistent problems could be handled by a strategy of precommitment or a strategy of time-consistency. Subsequently, Phelps and Pollak (1968), Pollak (1968) and Barro (1999) further considered time-inconsistent problems in different situations. Recently, Björk and Murgoci (2009), Wang and Forsyth (2011), Björk et al. (2010), Basak and Chabakauri (2010), Czichowsky (2010) and Zeng and Li (2011) paid much attention to time-inconsistent stochastic control problems and aimed at deriving the optimal time-consistent strategies.

As far as we know, there is little work in the literature on the optimal portfolio strategies under the mean-variance criterion with SV models, and only Zeng and Li (2011) have considered the optimal time-consistent investment and reinsurance strategies for insurers under the mean-variance criterion with the Black–Scholes model. Besides, Heston's SV model is very popular for option pricing. Therefore, in this paper, we study the optimal time-consistent investment and reinsurance strategies for insurers with Heston's SV model. Specifically, the surplus processes of insurers are assumed to follow a Brownian motion with drift; the financial market consists of one risk-free asset and one risky asset whose price satisfies Heston's SV model. We first formulate a general problem, and provide the corresponding verification theorem. Second, we derive the explicit closed-form expressions of the optimal time-consistent strategies and the corresponding value functions for the mean-variance problem without precommitment under two cases: the investment–reinsurance case and the investment-only case. In the first case, insurers are allowed to purchase proportional reinsurance, acquire new business and invest in the financial market. In the second case, insurers are only allowed to invest in the financial market but not to purchase proportional reinsurance or acquire new business. At the end, some economic implications of our results and sensitivity analysis with numerical illustrations are presented. This paper has three main contributions: (i) the optimal portfolio problem under the mean-variance criterion with SV models is studied; (ii) the time-inconsistent investment and reinsurance problems with SV models is considered, and the closed-form expressions of the optimal time-consistent strategies and the corresponding value functions are derived; (iii) a verification theorem for general problems, including the mean-variance problem without precommitment, is provided.

The remainder of this paper is organized as follows. The model and assumptions are described in Section 2. In Section 3, a general time-inconsistent problem is formulated, and the corresponding verification theorem is provided. In Section 4, the closed-form solution for the mean-variance problem without precommitment is derived in two scenarios: the investment–reinsurance case and the investment-only case. Some economic implications and sensitivity analysis with numerical illustrations are presented in Section 5, and Section 6 presents our conclusions.

2. Model and assumptions

In this paper, we assume that there are no transaction costs or taxes in the financial market or the insurance market, and trading can be continuous. All stochastic processes introduced below are assumed to be well-defined and adapted processes in a given filtered complete probability space $(\Omega, \mathcal{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathcal{P})$, where T is a positive finite constant representing the time horizon; $\{\mathscr{F}_t\}_{t\in[0,T]}$ is a filtration; each $\mathscr{F}_t$ can be interpreted as the information available at time t, and any decision made at time t is based upon such information.

2.1. Surplus process

We assume that the insurer's surplus process is modeled by a diffusion approximation (DA) model:

$$\mathrm{d}R_0(t) = \mu_0\,\mathrm{d}t + \sigma_0\,\mathrm{d}W_0(t), \tag{1}$$

where μ_0 represents the premium return rate of the insurer; $\sigma_0 > 0$ can be regarded as the volatility of the insurer's surplus; $\{W_0(t)\}$ is a one dimensional standard Brownian motion. Readers interested in how to derive the DA model are referred to Grandll (1991). The DA model (1) works well for large insurance portfolios since each claim is relatively small compared to the size of surplus, and this model has been widely used in the literature, for example, Browne (1995), Promislow and Young (2005), Gerber and Shiu (2006), Bai and Guo (2008), Chen et al. (2010), and so on.

We assume that the insurer can control its insurance risk by purchasing proportional reinsurance or acquiring new business, for example, by acting as a reinsurer of other insurers (see Bäuerle (2005)). For each $t \in [0, T]$, the proportional reinsurance/new business level is denoted by the value of risk exposure $a(t) \in [0, +\infty)$. When $a(t) \in [0, 1]$, it corresponds to a proportional reinsurance cover; in this case, the cedent should divert part of the premium to the reinsurer at the rate of $(1 - a(t))\eta$, where $\eta \geq \mu_0$ is the premium return rate of the reinsurer; meanwhile, the insurer pays $100a(t)$% while the reinsurer pays the rest $100(1-a(t))$% for each claim occurring at time t. When $a(t) \in (1, +\infty)$, it corresponds to acquiring new business. The process of risk exposure $\{a(t) : t \in [0, T]\}$ is called the reinsurance strategy, and the DA dynamics for the surplus process associated with such a reinsurance strategy $\{a(t) : t \in [0, T]\}$ is given by

$$\mathrm{d}R(t) = [\mu_0 - (1 - a(t))\eta]\,\mathrm{d}t + \sigma_0 a(t)\,\mathrm{d}W_0(t). \tag{2}$$

2.2. Financial market

We assume that the financial market consists of one risk-free asset (e.g. a bond or a bank account) and one risky asset (e.g. a stock); the price process $S_0(t)$ of the risk-free asset evolves according to the ordinary differential equation (ODE)

$$\mathrm{d}S_0(t) = r_0 S_0(t)\,\mathrm{d}t, \quad S_0(0) = s_0 > 0, \tag{3}$$

where $r_0 > 0$ represents the risk-free interest rate; the price processes $S_1(t)$ of the risky asset follows Heston's SV model

$$\begin{cases} \mathrm{d}S_1(t) = S_1(t)\left[(r_0 + \lambda L(t))\,\mathrm{d}t + \sqrt{L(t)}\,\mathrm{d}W_1(t)\right], \\ \quad S_1(0) = s_1 > 0, \\ \mathrm{d}L(t) = k(\theta - L(t))\,\mathrm{d}t + \sigma\sqrt{L(t)}\,\mathrm{d}W_2(t), \\ \quad L(0) = l_0 > 0, \end{cases} \tag{4}$$