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Insurance: Mathematics and Economics 31 (2002) 35–69

INSURANCE MATHEMATICS & ECONOMICS

www.elsevier.com/locate/econbase

Optimal investment strategies and risk measures in defined contribution pension schemes

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Received 1 November 2001; received in revised form 1 March 2002; accepted 8 April 2002

Abstract

In this paper, we derive a formula for the optimal investment allocation (derived from a dynamic programming approach) in a defined contribution (DC) pension scheme whose fund is invested in n assets. We then analyse the particular case of $n = 2$ (where we consider the presence in the market of a high-risk and a low-risk asset whose returns are correlated) and study the investment allocation and the downside risk faced by the retiring member of the DC scheme, where optimal investment strategies have been adopted. The behaviour of the optimal investment strategy is analysed when changing the disutility function and the correlation between the assets. Three different risk measures are considered in analysing the final net replacement ratios achieved by the member: the probability of failing the target, the mean shortfall and a value at risk (VaR) measure. The replacement ratios encompass the financial and annuitisation risks faced by the retiree. We consider the relationship between the risk aversion of the member and these different risk measures in order to understand better the choices confronting different categories of scheme member. We also consider the sensitivity of the results to the level of the correlation coefficient.

Keywords: Defined contribution pension scheme; Optimal investment; Downside risk

1. Introduction

In this paper we first derive and analyse the optimal investment strategy for a defined contribution (DC) pension scheme whose fund is invested in n assets, and then consider the special case of two assets and study the optimal investment strategy behaviour and the downside risk (in terms of the net replacement ratio achieved at retirement) faced by the member of the scheme, thereby extending the model introduced in Vigna and Haberman (2001).

The extensions introduced are three-fold:

1. we consider n assets instead of two;
2. we now consider assets which are correlated with each other;
3. we generalise the disutility function in such a way that deviations of the fund above the target are not penalised to the same degree as deviations below and the risk profile of the individual is taken into consideration.

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PII: S0167-6687(02)00128-2

For the case of the $n = 2$, the downside risk has been studied by examining three risk measures: the probability of failing the target, the mean shortfall and the value at risk (VaR) measure at three different confidence levels (1, 5 and 10%).

The annuity risk faced by the member has been analysed through these risk measures by comparing the results relative to the net replacement ratio both in the case of a fixed conversion factor and in the case of a random conversion factor, which depends on the prevailing yield on the low-risk asset.

2. The model

We consider a DC pension scheme with n-asset portfolio. The forces of interest corresponding to the investment returns of the n assets are assumed to be normally distributed and correlated at any time with a given variance–covariance matrix.

Contributions are paid in advance every year as a fixed proportion of the salary of the scheme's member. Taxation, expenses and decrements other than retirement are not taken into consideration. The scheme member is assumed to join the scheme at time $t = 0$ and contribute until retirement at time $t = N$, which is a time point that is fixed in advance.

The model is presented in discrete time and we assume that the portfolio is reallocated every year between the n assets, depending on the past history of the market returns and on the current size of the fund, which is compared to an a priori target. We then find the optimal investment allocation every year that minimises the deviations of the fund from these corresponding targets. We assume that there are no real salary increases and that for simplicity the salary is 1 each year.

The fund at time $t + 1$ is given by the following equation:

$$f_{t+1} = (f_t + c)\left[\sum_{i=1}^{n-1} y_t^i (\mathrm{e}^{X_t^i} - \mathrm{e}^{X_t^n}) + \mathrm{e}^{X_t^n}\right], \quad t = 0, 1, \ldots, N-1, \tag{2.1}$$

where f_t is the fund level at time t, c the contribution rate, y_t^i the proportion of the portfolio invested in the ith asset during year $[t, t+1]$, $i = 1, 2, \ldots, n-1$, so that the proportion invested in the nth asset is $1 - \sum_{t=1}^{n-1} y_t^i$ and X_t^i is the force of interest of ith asset in year $[t, t+1]$, assumed to be constant over year $[t, t+1]$, $i = 1, 2, \ldots, n$.

For fixed i, the sequences $\{X_t^i\}_{t=0,1,\ldots,N-1}$ are assumed to be IID with a normal distribution, while the correlation structure for the annual forces of interest X_t^i and X_t^j is given by the variance–covariance matrix, which is assumed to be constant for any t.

Therefore,

$$X_t^i \approx N(\mu_i, \sigma_i^2) \quad \text{for } t = 0, 1, \ldots, N-1,$$

where we assume, without loss of generality, that

$$\mu_1 > \mu_2 > \cdots > \mu_n \quad \text{and} \quad \sigma_1^2 > \sigma_2^2 > \cdots > \sigma_n^2.$$

3. The problem

3.1. Formulation of the problem

We define the "cost" incurred by the fund at time t as follows:

$$C_t = (F_t - f_t)^2 + \alpha(F_t - f_t), \quad t = 0, 1, \ldots, N-1, \tag{3.1}$$

$$C_N = \theta[(F_N - f_N)^2 + \alpha(F_N - f_N)], \quad t = N \tag{3.2}$$