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Optimal investment strategy to minimize the ruin probability of an insurance company under borrowing constraints

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ABSTRACT

We consider that the surplus of an insurance company follows a Cramér–Lundberg process. The management has the possibility of investing part of the surplus in a risky asset. We consider that the risky asset is a stock whose price process is a geometric Brownian motion. Our aim is to find a dynamic choice of the investment policy which minimizes the ruin probability of the company. We impose that the ratio between the amount invested in the risky asset and the surplus should be smaller than a given positive bound a. For instance the case $a = 1$ means that the management cannot borrow money to buy stocks.

[Hipp, C., Plum, M., 2000. Optimal investment for insurers. Insurance: Mathematics and Economics 27, 215–228] and [Schmidli, H., 2002. On minimizing the ruin probability by investment and reinsurance. Ann. Appl. Probab. 12, 890–907] solved this problem without borrowing constraints. They found that the ratio between the amount invested in the risky asset and the surplus goes to infinity as the surplus approaches zero, so the optimal strategies of the constrained and unconstrained problems never coincide.

We characterize the optimal value function as the classical solution of the associated Hamilton–Jacobi–Bellman equation. This equation is a second-order non-linear integro-differential equation. We obtain numerical solutions for some claim-size distributions and compare our results with those of the unconstrained case.

1. Introduction

An important issue in actuarial theory is to study the ruin probability of an insurance company when the management has the possibility of investing in the financial market. Paulsen and Gjessing (1997) and Paulsen (1998) treated this problem assuming that all the surplus is invested in the risky asset. Frovola et al. (2002) studied the asymptotic behavior of the ruin probability for light tails in the case where a constant proportion of the surplus is invested in the risky asset (stocks).

The problem of finding the optimal dynamic investment strategy which minimizes the ruin probability was first studied by Browne (1995) under the assumptions that the uncontrolled surplus follows a Brownian motion (diffusion approximation) and that the financial market follows a classical Black–Scholes model consisting on a risk-free asset (bond) and a risky asset (stock).

Hipp and Plum (2000) found the investment strategy which minimizes the ruin probability modeling the aggregate claim amount as in the classical risk model. They found that, in the optimal investment strategy, the management should borrow money to invest in the risky asset; in this optimal strategy, the ratio between the amount invested in the risky asset and the surplus goes to infinity as the surplus approaches zero. In fact, there are examples where, under the optimal investment strategy, the company should be in debt no matter the surplus (see Example 6.3). In this paper we treat the same problem but imposing borrowing constraints; namely we impose that the ratio between the amount invested in the risky asset and the surplus should be smaller than a given positive bound a. For instance the case $a = 1$ means that the management has the possibility to invest in the risky asset any proportion of the surplus but cannot borrow money to buy stocks.

Schmidli (2002) solved a related problem, he found the optimal investment strategy and the optimal proportional reinsurance policy which minimizes the ruin probability, with no borrowing constraints.

In the setting of the diffusion approximation, Promislow and Young (2005) and Luo (2008) found the optimal investment strategy and the optimal proportional reinsurance policy which minimizes the ruin probability under different borrowing constraints. Vila and Zariphopoulou (1997) and Bayraktar and Young (2007) studied other optimal control problems with borrowing constraints.

In this paper, we first obtain the Hamilton–Jacobi–Bellman equation associated to the optimization problem, which is a

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non-linear degenerate second-order integro-differential equation. Then we construct a weak solution of the HJB equation as a fixed point of a non-linear integral operator. We study the regularity of the weak solution and prove that it is twice continuously differentiable and so, a classical solution of the HJB equation. Finally, we use Itô's formula and martingale techniques to prove that the optimal survival probability is a multiple of the solution obtained previously. Moreover, we obtain the optimal constrained strategy and show that it depends only on the current surplus. We also prove that for small surpluses, the optimal policy is to invest the maximum allowed in the risky asset. In order to obtain the optimal survival probability, we construct via a fixed-point operator, the survival probability corresponding to invest in the risky asset a fixed proportion of the surplus and proved that it is a classical solution of the corresponding integro-differential equation.

We include numerical examples comparing the optimal survival probabilities and the optimal investment strategies in the unconstrained and constrained cases.

This paper is organized as follows: In Section 2, we state the optimization problem and the HJB equation. In Sections 3 and 4, we construct via a fixed-point operator a weak solution of the HJB equation and show that it is a classical solution. In Section 5, we use the solution obtained in Section 3 to obtain the optimal ruin probability and describe the optimal investment strategy. In Section 6, we show some numerical examples.

2. Formulation of the optimization problem

We assume that the financial market is described as a classical Black–Scholes model where we have a risk-free asset with price process B_t and a risky asset with price process S_t satisfying

$$\begin{cases} dB_t = r_0 B_t dt \\ dS_t = rS_t dt + \sigma S_t dW_t \end{cases}$$

where W_t is a standard Brownian motion. In this paper, we consider that all the monetary quantities are discounted by inflation, so we have that $r_0 = 0$.

We consider that the surplus X_t of the insurance company, without investments, follows the classical Cramér–Lundberg process, that is

$$X_t = x + (p_0 - d)t - \sum_{i=1}^{N_t} U_i$$

where x is the initial surplus, p_0 is the premium rate, d is the rate of dividend payments, N_t is a Poisson process with intensity $\beta > 0$, and the claim sizes U_i are i.i.d. random variables with distribution F with finite mean. We assume that the random variables U_i and the processes N_t and W_t are independent. We also assume, as in Hipp and Plum (2000) and Schmidli (2002), that F has a bounded density.

Let Ω be the set of càdlag paths and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be the complete probability space generated by the process (X_t, W_t). For a given upper bound $a > 0$, a control strategy is a process $(\gamma_t)_{t\geq 0}$ where $\gamma_t \in [0, a]$ is the ratio between the amount invested in the risky asset and the surplus at time t. We say that a control strategy is *admissible* if γ_t is predictable for every $t \geq 0$. For instance, $a = 1$ means that we are not allowing short-selling of stocks or to borrow money to buy stocks.

We denote by Π_x the set of all the admissible control strategies with initial surplus x. Given an admissible strategy $\gamma \in \Pi_x$, the controlled risk process X_t^γ is given by

$$X_t^\gamma = x + pt + r\int_0^t X_s^\gamma \gamma_s ds + \sigma \int_0^t X_s^\gamma \gamma_s dW_s - \sum_{i=1}^{N_t} U_i, \tag{2.1}$$

where $p = p_0 - d$.

We define the corresponding ruin time τ^γ of the company as

$$\tau^\gamma = \inf\{t \geq 0 : X_t^\gamma < 0\} \tag{2.2}$$

and the corresponding survival probability

$$\delta_\gamma(x) = 1 - P(\tau^\gamma < \infty). \tag{2.3}$$

We assume that $p > 0$.

The optimal survival probability is defined as

$$\delta(x) = \sup\{\delta_\gamma(x) \text{ with } \gamma \in \Pi_x\} \tag{2.4}$$

for all $x \geq 0$. To simplify notation we define $\delta(x) = 0$ for $x < 0$.

A Hamilton–Jacobi–Bellman equation can be associated to this optimization problem in the following way: if δ is a twice continuously differentiable function then it will be a solution of the integro-differential equation

$$\mathcal{L}^*(\delta)(x) = 0, x \geq 0, \tag{2.5}$$

where

$$\mathcal{L}^*(\delta)(x) = \sup_{\gamma\in[0,a]} \mathcal{L}_\gamma(\delta)(x), \tag{2.6}$$

$$\mathcal{L}_\gamma(\delta)(x) = \frac{\sigma^2\gamma^2 x^2}{2}\delta''(x) + (p + \gamma rx)\,\delta'(x) - M(\delta)(x) \tag{2.7}$$

and

$$M(\delta)(x) = \beta\delta(x) - \beta\int_0^x \delta(x-\alpha)dF(\alpha). \tag{2.8}$$

In addition, δ satisfies the boundary condition $\lim_{x\to\infty}\delta(x) = 1$.

Remark 2.1. It is clear that δ is increasing. We will prove in Proposition 5.4 that p positive implies that there exists a constant admissible strategy $\gamma_0 \in \Pi_0$ such that $\delta_{\gamma_0}(0) > 0$ and so $\delta(0) > 0$.

3. Weak solutions of $\mathcal{L}^* = 0$

In this section we construct via a fixed-point operator a solution W of (2.5) with boundary condition $W(0) = 1$ in a *weak* sense. In the next section we will prove that this solution is twice continuously differentiable and so it is the unique classical solution of this equation with boundary condition $W(0) = 1$. Heuristically, the reason why there exists a unique solution of this second-order integro-differential equation with only a boundary condition at $x = 0$ is that the ellipticity of the operator $\mathcal{L}^*$ degenerates at this point.

We would like to point out that in the unconstrained case (see Hipp and Plum (2000)), since the survival probability δ is concave and $\mathcal{L}_\gamma(\delta)(x)$ is a quadratic function on γ, the supremum is attained at the vertex. So the authors can get rid of γ and construct the solution as a fixed point of a contraction operator without supremum. In their case the regularity of the solution follows from the fixed-point theorem.

In the constrained case, the supremum of $\mathcal{L}_\gamma$ could also be attained at $\gamma = a$ or $\gamma = 0$ and so the fixed-point operator is more complex because it involves a supremum. Then, we cannot obtain as a consequence of the fixed-point theorem enough regularity to have a classical solution.

An operator similar to $\mathcal{L}^*$ is involved in the HJB equation associated to the problem of maximizing the cumulative expected dividend pay-outs in an insurance company when the management has the possibility of investing part of the surplus in a risky asset. See Azcue and Muler (submitted for publication).

In the next proposition we show that any twice continuously differentiable solution of the Eq. (2.5) is a fixed point of some