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Optimal investment strategy for annuity contracts under the constant elasticity of variance (CEV) model

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ABSTRACT

This paper focuses on the constant elasticity of variance (CEV) model for studying the optimal investment strategy before and after retirement in a defined contribution pension plan where benefits are paid under the form of annuities; annuities are supposed to be guaranteed during a certain fixed period of time. Using Legendre transform, dual theory and variable change technique, we derive the explicit solutions for the power and exponential utility functions in two different periods (before and after retirement). Each solution contains a modified factor which reflects an investor's decision to hedge the volatility risk. In order to investigate the influence of the modified factor on the optimal strategy, we analyze the property of the modified factor. The results show that the dynamic behavior of the modified factor for the power utility mainly depends on the time and the investor's risk aversion coefficient, whereas it only depends on the time in the exponential case.

1. Introduction

There are two main ways to manage a pension fund, one is defined benefit (hereafter DB) schemes, where benefits are fixed in advance by the sponsor and contributions are initially set and subsequently adjusted in order to maintain the fund in balance. The other is defined contribution (hereafter DC) schemes, where the contributions are fixed and benefits depend on the returns on the fund's portfolio. The main difference between DB and DC pension schemes is the way in which the financial risk is treated. In DB plans, the financial risk is borne by the sponsors of the scheme. In DC plans, the financial risk is borne by the contributors. In particular, the financial risk can be split into two parts: investment risk during the before retirement period, and annuity risk focused on the retirement period. Recently, due to the demographic evolution and the development of the equity market, DC schemes have become popular in the global pension market.

The current actuarial literature about the financial risk in DC pension schemes is quite rich. The financial risk in DC schemes in the accumulation phase is considered, among others, by Blake et al. (2001), Booth and Yakoubov (2000), Boulier et al. (2001), Haberman and Vigna (2002), Deelstra et al. (2004). The financial risk in the distribution phase of defined contribution pension schemes has been dealt with in many papers, see, e.g., Albrecht and Maurer (2002), Blake et al. (2003) and Gerrard et al. (2004). In addition, Devolder et al. (2003) and Xiao et al. (2007) analyzed the financial risk both in the accumulation phase and in the distribution phase for a defined contribution pension scheme. However, most of these researches generally supposed the risky asset price dynamics driven by a geometric Brownian motion (GBM). This implies that the volatility of the risky asset price is only a constant without considering the time-dependence in the financial market.

The purpose of this paper is to concentrate on the constant elasticity of variance (CEV) model for studying the optimal investment strategy for the before and after retirement periods in a DC pension scheme. The CEV model with stochastic volatility is a natural extension of the GBM. It was originally proposed by Cox and Ross (1976) as an alternative diffusion process for European option pricing. Compared with the GBM, the advantage of the CEV model is that the volatility has correlation with the risky asset price and can explain the empirical bias exhibited by the Black and Scholes (1973) model, such as volatility smile. The CEV model was usually applied to analyzing the options and an asset pricing formula, see, e.g., MacBeth and Merville (1980), Schroder (1989), Cox (1996), Davydov and Linetsky (2001), Detemple and Tian (2002), Jones (2003) and Hsu et al. (2008). Recently, Xiao et al. (2007) began to apply the CEV model to an annuity contract

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and derived a dual solution for the logarithm utility via Legendre transform and dual theory. However, they failed to find the solutions for the other utility functions. This happened because it is difficult to characterize the solution structure of the dual variable in the three space dimensions. So far, apart from logarithm utility function, the application of the CEV model to a pension fund with other utility functions has not been reported in academic articles.

In this paper, we consider the CEV-extended annuity contracts with a power (CRRA) and exponential (CARA) utility functions. Applying the method of stochastic optimal control, Legendre transform and dual theory (cf. Jonsson and Sircar (2002), Xiao et al. (2007)), we derive a linear partial differential equation (PDE) for the value function of the optimization problem. However, under the given utility function, we notice that the PDE's coefficient variables are closely correlated, and then it is very hard to characterize the solution structure. Therefore, we use the variable change technique suggested by Cox (1996) in the derivation of the CEV option price to transform it into a simple PDE and obtain the explicit solutions for the two phases: before and after retirement.

Each solution contains a modified factor, which represents a supplementary part resulted from the changes of the volatility under a CEV model. In other words, the modified factor reflects an investor's decision to hedge the volatility risk. In addition, we find that the modified factor is equal to one as the elasticity coefficient $\beta = 0$. In this case, the optimal strategy will reduce to the relative result provided by Devolder et al. (2003). Therefore, the optimal strategy we derive can be considered as an extension result of Devolder et al. (2003). Moreover, we find that our solution for a power utility is just the result of Xiao et al. (2007) as the risk aversion coefficient p tends to zero.

Finally, in order to have a better understanding of the impact of the modified factor on the optimal strategy, we analyze the properties of the modified factor. We find that the dynamic behavior of the modified factor for a CRRA utility mainly depends on the time and the investor's risk aversion coefficient, whereas it only depends on the time in the CARA case. Moreover, the modified factor for a CARA utility is a monotone increasing function with respect to time t and the value is less than or equal to one. This implies that the suggestion of the modified factor for an investor is to invest initially smaller proportions of wealth in stock and later increase the proportions significantly. In the case of a CRRA utility, we obtain two conclusions. One is the same as that in the CARA case if the investor's risk aversion coefficient is less than zero. The other is actually the complete opposite as the investor's risk aversion coefficient lies between zero and one. This situation indicates that the modified factor advises the investor to invest a larger proportion of wealth in stock at the initial time and steadily reduce the proportion as time passes.

The rest of the paper is organized as follows. In Section 2, we introduce the financial market and propose the optimization problems for the two periods: before retirement and after retirement. In Section 3, we provide the general framework to solve the optimal problem for the two periods. In Sections 4 and 5, we derive two explicit solutions for the two periods, respectively. Conclusions are given in Section 6.

2. The model

In this section, we introduce the financial market and propose the optimization problems for the two periods: before retirement and after retirement.

2.1. The financial market

We consider a financial market consisting of a risk-free asset and a single risky asset. We denote the price of the risk-free asset (i.e. the bank account) at time t by B_t, which evolves according to the following formula:

$$\mathrm{d}B_t = rB_t\mathrm{d}t, \tag{2.1}$$

where r is a constant rate of interest.

We denote the price of the risky asset (hereinafter called 'stock') at time t by S_t, which is described by the CEV model (cf. Davydov and Linetsky (2001), Jones (2003), Hsu et al. (2008)):

$$\frac{\mathrm{d}S_t}{S_t} = \mu\mathrm{d}t + kS_t^{\beta}\mathrm{d}W_t, \tag{2.2}$$

where μ ($\mu > r$) is an expected instantaneous rate of return of the stock. k and β are constant parameters, and β satisfies the general condition $\beta < 0$. kS_t^{β} is the instantaneous volatility. $\{W_t : t \geq 0\}$ is a standard Brownian motion defined on a probability space $(\Omega, \mathscr{F}, P)$ endowed with an augmented filtration $\mathscr{F} = \{\mathscr{F}_t\}$ generated by the Brownian motion.

Remark 1. Note that if the elasticity parameter $\beta = 0$ in Eq. (2.2), then it reduces to a GBM. If $\beta < 0$, the instantaneous volatility kS_t^{β} increases as the stock price decreases, and can generate a distribution with a fatter left tail. If $\beta > 0$, the situation is reversed and unrealistic.

2.2. The optimization program

Once the assets available to the pension investor have been described, we can model the pension fund investment. The problem is to find the optimal investment strategy for the assets over the whole life of a participant in the plan.

Following Devolder et al. (2003) and Xiao et al. (2007), the pension liabilities after retirement are supposed to be paid in the form of an annuity whose level is guaranteed by the insurer. We will split the problem into two periods, before and after retirement

2.2.1. Period before retirement

During the period before retirement ($t \in [0, T]$), the contributions can be invested in a risk-free asset and a stock. The purpose is to maximize the utility of the final wealth at retirement.

Let V_t denote the pension wealth at time $t \in [0, T]$. The dynamics of wealth are described by:

$$\mathrm{d}V_t = [V_t(\pi_t(\mu - r) + r) + c]\mathrm{d}t + \pi_t kS_t^{\beta}V_t\mathrm{d}W_t, \qquad V_0 = M, \tag{2.3}$$

where π_t denotes the proportion of wealth invested in the stock at time t, the contribution rate no-loss generality is a constant c and the wage is one unit. M stands for the initial wealth.

Under the wealth process denoted by (2.3), the investor looks for a strategy π_t^* maximizing the utility function:

$$\max_{\{\pi_t\}} E(U(V_T)), \tag{2.4}$$

where $U(\cdot)$ is an increasing concave utility function and satisfies the Inada conditions $u'(+\infty) = 0$ and $u'(0) = +\infty$.

2.2.2. Period after retirement

According to Devolder et al. (2003) and Xiao et al. (2007), at retirement ($t = T$), the fund will purchase a paid-up annuity; the purchase rate is computed on a predetermined interest rate. The part of the fund used to purchase an annuity of N periods is denoted as D, where $D \leq V_T$. The surplus at the end of the fixed period can be used again in a similar way or paid back to the participants. The contributions benefit to pay between T and $T + N$ is given by

$$L = D/\bar{a}_{\overline{N}|},$$

where $\bar{a}_{\overline{N}|} = \left(1 - \mathrm{e}^{-\delta N}/\delta\right)$, δ is a continuous technical rate.