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Market selection of constant proportions investment strategies in continuous time

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ABSTRACT

This paper studies the wealth dynamics of investors holding self-financing portfolios in a continuous-time model of a financial market. Asset prices are endogenously determined by market clearing. We derive results on the asymptotic dynamics of the wealth distribution and asset prices for constant proportions investment strategies. This study is the first step towards a theory of continuous-time asset pricing that combines concepts from mathematical finance and economics by drawing on evolutionary ideas.

1. Introduction

This paper aims at developing a theory of asset pricing that is based on the market interaction of traders in a continuous-time mathematical finance framework. While mathematical finance has offered deep insights in the dynamics of portfolio payoffs under the assumption of an exogenous price process, economists prefer—in a market context—equating demand and supply through an endogenous price mechanism. This market interaction of investors plays a central role in our approach; it is modeled through the introduction of endogenous prices (driven by demand and supply) in the classical mathematical finance model. Randomness stems from exogenous asset payoff processes (dividends) and variation in traders' behavior.

Our analysis focuses on the survival and extinction of investment strategies which is defined through the asymptotic outcome of the wealth dynamics. This evolutionary view to financial markets is explored in discrete-time models by Evstigneev et al. (2006, 2008, 2002) (see also their survey, Evstigneev et al., 2009). In the present paper these ideas are used to develop an evolutionary finance model in continuous time. This approach incorporates market interaction of traders into the workhorse model of mathematical finance. The continuous-time setting overcomes the problem of a priori setting a frequency of trade. It also defines a benchmark for the specification of discrete-time models in which the wealth dynamics is consistent over different trading frequencies (see Palczewski and Schenk-Hoppé, forthcoming).

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The wealth dynamics in our continuous-time evolutionary finance model is described by a random dynamical system (Arnold, 1998). This system can be written as a non-linear differential equation with random coefficients that has not been studied before. Its asymptotic analysis requires the application of techniques that are new in this context. Specifically, we use the concept of a random fixed point (Schenk-Hoppé and Schmalfuss, 2001), the ergodic theory for Markov processes (Anderson, 1991) and the arcsine law for Markov chains (Freedman, 1963).

While the model is developed in the most general setting, our analysis is restricted to time-invariant investment strategies. These self-financing strategies prescribe to rebalance a portfolio so as to maintain constant proportions over time. This class is quite common in financial theory and practice, see, e.g., Browne (1998); Mulvey and Ziemba (1998) and Perold and Sharpe (1988). The assumption of time-invariance considerably reduces the level of mathematical difficulty. This allows to place the main emphasis on the novel ideas developed in this paper. Future research will aim to investigate the general, but mathematically more demanding, case of time-dependent strategies.

Our main result is the identification of a unique investment strategy λ^* that is asymptotically optimal in a market in which only time-invariant strategies are present. This strategy prescribes to divide wealth proportionally to the average relative dividend intensities of assets. We show that any other time-invariant investment strategy interacting in the market will become extinct by losing its wealth to the strategy λ^*. This finding has implications for asset pricing. If at least one investor follows the strategy λ^*, asset prices converge to a 'fundamental' value which (except for risk-free bonds) does not coincide with the usual valuation because it is based on averaging *relative* rather than absolute instantaneous dividend payments.

Our results provide the basis for a recommendation to portfolio managers targeting the well-documented success of pairs-trading strategies, see, e.g., Gatev et al. (2006). Investing according to the strategy λ^* generates excess returns (in the long-term) if there are assets whose relative valuation does not coincide with the benchmark. Indeed the proposed portfolio strategy is less risky than pairs-trading as it does not involve short positions.

The modeling approach presented here provides an alternative framework for portfolio optimization under the market impact of trades. Rather than postulating an exogenous price impact function for one large trader (e.g., Bank and Baum, 2004), our model provides an endogenous mechanism for the market impact of transactions (large and small). The impact increases with the size of a transaction but the precise market response depends on the wealth distribution across investors and their investment strategies.

The evolutionary approach to asset pricing presented here is related to asset pricing theories based on the notion of excess returns, e.g., Luenberger (1993) and Platen (2006), which can be traced back to the setting of betting markets studied by Kelly (1956). These theories, however, do not take into account market interaction of traders. Stochastic general equilibrium models (which do have endogenous prices), on the other hand, suffer from the interconnectedness of consumption and investment decisions. This feature precludes clear-cut results on the long-term dynamics of asset prices if markets are incomplete, see, e.g., Blume and Easley (2006).

The paper is organized in the following fashion. Section 2 introduces the model. Section 3 presents general selection results on the market dynamics. Section 4 considers the particular case of Markovian dividend intensities. Section 5 concludes. All proofs are collected in Appendix A.

2. The model

This section derives the general evolutionary finance model in continuous time.

Consider the following description of a financial market in continuous time which is based on the standard approach in mathematical finance (e.g., Björk, 2004 or Pliska, 1997). There are K assets (stocks) with cumulative dividend payments $D(t) = (D_1(t), \ldots, D_K(t)), t \geq 0$. Denote the price process, which will be specified later, by $S(t) = (S_1(t), \ldots, S_K(t))$. Each asset is in positive net supply of one. There are I investors. The portfolio (in numbers of physical units of assets) of investor i is denoted by $\theta^i(t) = (\theta^i_1(t), \ldots, \theta^i_K(t))$. His cumulative consumption process is given by $C^i(t)$. For a self-financing portfolio-consumption process $(\theta^i(t), C^i(t))$, the dynamics of investor i's wealth $V^i(t) = \sum_{k=1}^K \theta^i_k(t) S_k(t)$ is

$$dV^i(t) = \sum_{k=1}^{K} \theta^i_k(t-)(dS_k(t) + dD_k(t)) - dC^i(t), \tag{1}$$

where t denotes the left-hand limit. The self-financing property of the strategy means that changes in value can be attributed either to changes in asset prices, dividend income or consumption expenditure. If there is a jump in either price or dividend, the gain (or loss) is attributed to the investor holding the asset just prior to the occurrence of this event.

Each investor's portfolio can be written as

$$\theta^i_k(t) = \frac{\lambda^i_k(t) V^i(t)}{S_k(t)} \tag{2}$$

with some real-valued process $\lambda^i_k(t)$, provided $V^i(t) \neq 0$ and $S_k(t) \neq 0$. If investors hold only long positions, this representation is correct for $V^i(t) = 0$ by setting, for example, $\lambda^i_k(t) = 1/K, k = 1, \ldots, K$. The quantity $\lambda^i_k(t)$ can be interpreted as the trader's budget share allocated to the holding in asset k. We will refer to $\lambda^i(t) = (\lambda^i_1(t), \ldots, \lambda^i_K(t))$ as an *investment strategy*. It follows