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Constant elasticity of variance model for proportional reinsurance and investment strategies

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ABSTRACT

In our model, the insurer is allowed to buy reinsurance and invest in a risk-free asset and a risky asset. The claim process is assumed to follow a Brownian motion with drift, while the price process of the risky asset is described by the constant elasticity of variance (CEV) model. The Hamilton–Jacobi–Bellman (HJB) equation associated with the optimal reinsurance and investment strategies is established, and solutions are found for insurers with CRRA or CARRA utility.

1. Introduction

Reinsurance and investment are two of the most important issues for an insurer. Reinsurance ensures that the insurer's earnings remain relatively stable or to protect the insurer against potentially large losses (Cai et al., 2008), while investment enables that the insurer achieves its management objectives.

The focus of current research in reinsurance and investment has been on the optimal strategies for different reinsurance types and ruin probability reduction. For example, Cerqueti et al. (2009) presented a purely theoretical reinsurance model, where the reinsurance contract was assumed to be simultaneously of an excess-of-loss and of a proportional type. Taksar and Markussen (2003) used stochastic optimal control theory to determine the optimal reinsurance policy which minimizes the ruin probability of the cedent. Promislow and Young (2005) demonstrated the strategy for an insurer to minimize the probability of ruin through investing in a risky asset and purchasing quota-share reinsurance. Hipp and Vogt (2003) found the optimal dynamic unlimited excess-of-loss reinsurance strategy to minimize infinite time ruin probability, and proved the existence of a smooth solution of the corresponding Hamilton–Jacobi–Bellman equation.

Recently, Cao and Wan (2009) derived optimal strategies about how to purchase the proportional reinsurance and how to invest in the risk-free asset and risky asset by solving the corresponding Hamilton–Jacobi–Bellman equations.

In current relevant literature (e.g. Cao and Wan, 2009), the price of risky asset follows geometric Brownian motion (GBM). In this paper, we use a constant elasticity of variance (CEV) model for studying the optimal investment and reinsurance strategy. The motivation for using the CEV model is that GBM can be considered as a special case of the CEV model and thus looking at CEV is a natural extension to current literature. In addition, the CEV model has the ability of capturing the implied volatility skew, which allows us to see how an investor who is aware of skew differs from the one who is not. Moreover, compared with other stochastic volatility models involving volatility skew (e.g. Heston, 1993; Cox, 1996), the advantage of the CEV model is its analytical tractability. Thus, under a CEV model, we can analytically examine the influences of the volatility skew on the insurer's optimal strategies (Gao, 2009).

The CEV model was usually applied to calculating the theoretical price, sensitivities and implied volatility of options (e.g. Cox, 1996; Lo et al., 2000; Davydov and Linetsky, 2001; Detemple and Tian, 2002; Jones, 2003; Widdicks et al., 2005). Recently, Xiao et al. (2007) and Gao (2009) used the CEV model to the pension investment.

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2. The model

Following the framework of Promislow and Young (2005) and Cao and Wan (2009), we model the claim process C according to a Brownian motion with drift as follows:

$$dC(t) = adt - bdW^0(t) \tag{2.1}$$

where a and b are positive constants, $W^0(t)$ is a standard Brownian motion. Assume that the premium is paid continuously at the constant rate $c_0 = (1+\theta)a$ with safety loading $\theta > 0$.

When both reinsurance and investment are absent, the dynamics of the surplus is given by

$$dR(t) = c_0dt - dC(t).$$

Let $R : R^+ \to R^+$ denote the reinsurance function, and we assume that the insurer purchases the proportional reinsurance to reduce the underlying risk. Let Y denote the total claim, $R(Y)$ denote the reinsurance function, then $R(Y) = qY$,where q represents the proportion reinsured. The insurer (cedent) pays reinsurance premium continuously at the constant rate $c_1 = (1+\eta)aq$ with safety loading $\eta > \theta > 0$. The cedent will pay $100(1-q)\%$ of each claim while the rest $100q\%$ is paid by the reinsurer. Based on Eq. (2.1), the surplus process $R(t)$ is given by the dynamics

$$\begin{aligned} dR(t) &= c_0dt - (1-q)dC(t) - c_1dt \\ &= (\theta - \eta q)adt + b(1-q)dW^0(t). \end{aligned} \tag{2.2}$$

In addition to the reinsurance, we suppose that the insurer is allowed to invest its surplus in a financial market consisting of a risk-free asset (bond or bank account) and a risky asset. Let $S_0(t)$ denote the price process of the risk-free asset, specifically, it is given by

$$dS_0(t) = r_0S_0(t)dt, \quad r_0 > 0. \tag{2.3}$$

We assume that the price of the risky asset is a continuous time stochastic process. Cao and Wan (2009) used the geometric Brownian motion to simulate the price of the risky asset. As above mentioned, CEV has advantage over GBM, and we will apply CEV instead of GBM in the price of risky asset. Let S_1 denote the price process of the risky asset, which is described by the CEV model (Lo et al., 2000; Jones, 2003):

$$dS_1(t) = r_1S_1(t)dt + \sigma S_1^{\beta+1}(t)dW^1(t) \tag{2.4}$$

where r_1 is an expected instantaneous rate of return of the risky asset and satisfies the general condition $r_1 > r_0$. $\sigma S_1^{\beta}(t)$ is the instantaneous volatility, and β is the elasticity parameter and satisfies the general condition $\beta < 0$. $\{W^1(t) : t \geq 0\}$ is a standard Brownian motion defined on a complete probability space (Ω, F, P), where Ω is the real space, P is the probability measure. The filtration $F = \{F_t\}$ is a right continuous filtration of sigma–algebras on this space and denotes the information structure generated by Brownian motion. The processes $W^0(t)$ and $W^1(t)$ are assumed to be independent.

Remark 2.1. The price process of the risky asset reduces to a GBM if $\beta = 0$. If $\beta < 0$, the instantaneous volatility $\sigma_t = \sigma S_1^{\beta}$ increases as the stock price decreases, and can generate a distribution with a fatter left tail. If $\beta > 0$, the situation is reversed and unrealistic.

A strategy α_t is described by a dynamic process $(q(t), l(t))$, where $q(t)$ represents the proportion reinsured at time t, $l(t)$ represents the proportion invested into the risky asset at time t. Let $X(t)$ denote the resulting surplus process after incorporating strategy α_t into Eq. (2.2). The dynamics of $X(t)$ can be preserved as follows:

$$\begin{aligned} dX(t) &= [r_0(1-l(t))X(t) + (\theta - \eta q(t))a]dt + b(1 \\ &\quad - q(t))dW^0(t) + r_1l(t)X(t)dt + \sigma l(t)S_1^{\beta}(t)X(t)dW^1(t) \\ &= (r_0X(t) + (\theta - \eta q(t))a)dt + b(1-q(t))dW^0(t) \\ &\quad + (r_1 - r_0)l(t)X(t)dt + \sigma l(t)S_1^{\beta}(t)X(t)dW^1(t). \end{aligned} \tag{2.5}$$

A strategy α_t is said to be admissible when it is adapted to F_t and satisfies the borrowing or lending capability of the insurer. Denote the set of all admissible strategies by α.

3. Optimization problem

In this section, we are interested in maximizing the utility of the insurer's terminal wealth. The utility function $U(x)$ is typically increasing and concave ($u''(x) < 0$). For a strategy α_t, we define the utility attained by the insurer from state x at time t as

$$V_{\alpha_t}(t, s, x) = E_{\alpha_t}[U(X(T)) \mid X(t) = x, S(t) = s]. \tag{3.1}$$

Our goal is to find the optimal value function

$$V(t, s, x) = \sup_{\alpha_t \in \alpha} V_{\alpha_t}(t, s, x), \tag{3.2}$$

and the optimal strategy $\alpha^* = (q^*(t), l^*(t))$ such that $V_{\alpha^*}(t, s, x) = V(t, s, x)$.

The Hamilton–Jacobi–Bellman (HJB) equation associated with the optimization problem is

$$\begin{aligned} V_t + \max_{\alpha_t \in \alpha} \Big\{ & (r_0x + (\theta - \eta q)a + (r_1 - r_0)lx)V_x + r_1sV_s \\ & + \frac{1}{2}\sigma^2 s^{2\beta+2}V_{ss} + \sigma^2 lxs^{2\beta+1}V_{sx} + \frac{1}{2}(b^2(1-q(t))^2 \\ & + \sigma^2 l^2x^2s^{2\beta})V_{xx} \Big\} = 0. \end{aligned} \tag{3.3}$$

with $V(T, s, x) = U(x)$, where V_t, V_s, V_x, V_{sx}, V_{ss} and V_{xx} denote partial derivatives of first and second orders with respect to time, price of risky asset and wealth.

The first order maximizing conditions for the optimal strategies q^* and l^* are:

$$\begin{cases} -b^2(1-q^*)V_{xx} - a\eta V_x = 0 \\ (r_1 - r_0)xV_x + \sigma^2 xs^{2\beta+1}V_{sx} + \sigma^2 l^*x^2s^{2\beta}V_{xx} = 0. \end{cases} \tag{3.4}$$

We have

$$\begin{cases} q^* = 1 + \dfrac{a\eta}{b^2}\dfrac{V_x}{V_{xx}} \\ l^* = -\dfrac{(r_1 - r_0)xV_x + \sigma^2 xs^{2\beta+1}V_{sx}}{\sigma^2x^2s^{2\beta}V_{xx}} \\ \quad = -\dfrac{(r_1 - r_0)V_x + \sigma^2 s^{2\beta+1}V_{sx}}{\sigma^2xs^{2\beta}V_{xx}}. \end{cases} \tag{3.5}$$

Putting this in Eq. (3.3), we obtain a partial differential equation (PDE) for the value function V:

$$\begin{aligned} & V_t + (r_0x + (\theta - \eta)a)V_x - \frac{1}{2}\left(\frac{(r_1 - r_0)^2}{\sigma^2s^{2\beta}} + \frac{a^2\eta^2}{b^2}\right)\frac{V_x^2}{V_{xx}} \\ & - s(r_1 - r_0)\frac{V_xV_{sx}}{V_{xx}} + r_1sV_s + \frac{1}{2}\sigma^2s^{2\beta+2}V_{ss} \\ & - \frac{1}{2}\sigma^2s^{2\beta+2}\frac{V_{sx}^2}{V_{xx}} = 0 \end{aligned} \tag{3.6}$$

with $V(T, s, x) = U(x)$.

Here, we notice that the stochastic control problem described in the previous section has been transformed into a PDE. The problem is now to solve Eq. (3.6) for the value function V and replace it in Eq. (3.5) in order to obtain the optimal strategies.