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Irreversible investment with Cox–Ingersoll–Ross type mean reversion[☆]

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ABSTRACT

We solve a Dixit and Pindyck type irreversible investment problem in continuous time under the assumption that the project value follows a Cox–Ingersoll–Ross process. This setup works well for modeling foreign direct investment in the framework of real options, when the exchange rate is uncertain and the project value fixed in a foreign currency. We indicate how the solution qualitatively differs from the two classical cases: geometric Brownian motion and geometric mean reversion. Furthermore, we discuss analytical properties of the Cox–Ingersoll–Ross process and demonstrate potential advantages of this process as a model for the project value with regard to the classical ones.

1. Introduction

In the basic model of real option theory, an agent has a single irreversible investment opportunity that comes at a cost I and reaps reward $V(\tau)$ if made at time t, resulting in a payoff of $(V(\tau) - I)e^{-\rho\tau}$, discounted at rate ρ. The agent's problem is to choose the time at which to make the investment – i.e., to execute the real option – given that $V(t)$ is an observable stochastic process. Problems that fall into this category range from investment in real estate to problems of environmental economics and the reduction of greenhouse gas emissions. Real option theory originated with the work of Myers (1977); significant contributions have been made since (without aiming for completeness) by McDonald and Siegel (1986), Myers and Majd (1990), Dixit and Pindyck (1994), and recently by Henderson (2002), Henderson (2006) and Henderson and Hobson (2002). The purpose of this paper is to analyze and obtain explicit solutions for such a problem when the value of the underlying financial project follows a Cox–Ingersoll–Ross (CIR) process:

$$\mathrm{d}V(t) = \kappa\,(\theta - V(t))\,\mathrm{d}t + \sigma\sqrt{V(t)}\mathrm{d}W(t) \tag{1}$$

where $\kappa, \theta, \sigma > 0$ are constants and $W(t)$ is a standard Brownian motion. We make the standard assumption that $2\kappa\theta > \sigma^2$, which guarantees that the process remains strictly positive with a probability 1; see for example Alos and Ewald (2008). This process has been introduced into Finance by Cox et al. (1985) and with the exception of Carmona and Leon (2007) has never been used in the context of real option models. We will argue however, that for some applications in real options, it is a more suitable process than the standard ones.

In most of the early work, it was assumed that the underlying financial project followed a geometric Brownian motion (GBM). However, such a process lacks the mean reversion that affects many project values. The preferred mean reverting model is the geometric mean reversion (GMR):

$$\mathrm{d}V(t) = \kappa(\theta - V(t))V(t)\mathrm{d}t + \sigma V(t)\mathrm{d}W(t). \tag{2}$$

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However, it is absorbing at its lower bound of 0 and its property that the mean reversion speed is proportional to the project value, is not always reasonable. In contrast, the CIR process guarantees a positive project value, has a constant mean reversion speed and is not absorbing at 0. Furthermore, unlike GMR, the CIR process is truly centered around the mean reversion level, in the following sense. Taking expectations in (1) shows that the expectation $e(t) := \mathbb{E}(V(t))$ satisfies the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}e(t) = \kappa\left(\theta - e(t)\right). \tag{3}$$

The solution is

$$e(t) = \theta + (V_0 - \theta)\mathrm{e}^{-\kappa t} \tag{4}$$

and therefore $\lim_{t\to\infty} e(t) = \theta$. As shown in Ewald and Yang (2007), the distribution of the GMR process is instead shifted away from the mean reversion level: $\lim_{t\to\infty} \mathrm{e}^{gmr}(t) = \theta - \frac{\sigma^2}{\kappa}$. Taking GMR as an underlying model, the mean reversion parameter θ can hence not be interpreted as an average price/cost in the long run.

Metcalf and Hasset (1995) present an empirical study which is mainly pro GMR as an underlying in a real option model, as compared to GBM. However, Metcalf and Hasset reflect on commodity prices only. In the context of interest and currency exchange rates, GMR can be easily rejected by means of data. Here CIR takes an eminent role and empirical studies underline this. We do not wish to argue at this point, that CIR is a more realistic model for commodity prices than GMR, but present the following hypothetical example of foreign direct investment. An agent wishes to (irreversibly) invest in a project in a foreign country. The project itself is relatively safe, so we assume its value in foreign currency is fixed at level p. However, the agent thinks in terms of domestic currency, and therefore the price he/she needs to pay at the time of investment is $pV(\tau)$ where $V(\tau)$ is the exchange rate at time τ. This leads directly to an irreversible investment model, with CIR as the underlying.

Carmona and Leon (2007) have also considered CIR in an irreversible investment context. In their exposition the project value is fixed as above and the agent has to reflect on stochastic interest rates, which are assumed to be CIR. Since the present value of the project depends on accumulated rather than spot interest rates (as opposed to the foreign currency example above) the underlying in Carmona and Leon is basically an accumulated CIR $\int_0^\tau V(s)\mathrm{d}s$ leading to an affine term structure, which makes their analysis significantly different from ours. We find that both approaches have their justification, but in neither of them would GMR (not to mention GBM) make any sense at all.

The article is organized as follows: We briefly summarize the main points of irreversible investment theory in continuous time in the next section and present the results in the two classical cases (GBM and GMR). In Section 3 we present our solution for the case of CIR and undertake a comparative analysis with respect to the classical cases in Section 4. This also includes various numerical experiments and graphical outputs. Conclusions are summarized in Section 5.

2. Irreversible investment: The classical cases

We briefly review the analysis presented in Dixit and Pindyck (1994). Assuming that the project value follows the dynamic

$$\mathrm{d}V(t) = \alpha(V(t))\mathrm{d}t + \sigma(V(t))\mathrm{d}W(t) \tag{5}$$

with $\alpha(v)$ and $\sigma(v)$ sufficiently regular functions and $W(t)$ a Brownian motion, it is shown that the value function $F(V)$ of the optimal stopping problem

$$F(V) = \max_{\tau}\left(\mathbb{E}\left[(V(\tau) - I)\,\mathrm{e}^{-\rho\tau}\right]\middle|\, V(0) = V\right) \tag{6}$$

satisfies the following Hamilton–Jacobi–Bellman equation

$$F'(V)\alpha(V) + \frac{1}{2}F''(V)\sigma^2(V) - \rho F(V) = 0 \tag{7}$$

subject to the following two conditions:

$$F(V^*) = V^* - I \tag{8}$$

$$F'(V^*) = 1. \tag{9}$$

Eq. (8) resp. (9) are referred to as value matching resp. smooth pasting condition. The value V^* is called the threshold or exercise value and the optimal stopping time is given by

$$\tau^* = \inf_{t\geq 0}\left(V(t) = V^*\right). \tag{10}$$

Eq. (7) represents a second order ODE, and its solution is entirely determined by two independent boundary conditions, which in principle would be provided by (8) and (9). However, here V^* is also unknown and a third condition is required to guarantee uniqueness of this solution. Dixit and Pindyck suppose the condition

$$F(0) = 0. \tag{11}$$

In the classical cases, where $V(t)$ is assumed to be either GBM or GMR and 0 is an absorbing state for the project value, this makes sense, because then the option expires as worthless if the project value falls to 0. We will see that in the CIR case, where 0 is not an absorbing state, condition (11) does not make sense and has in fact to be replaced by

$$F(0) < \infty. \tag{12}$$

The analysis of the classical cases shows that replacing (11) with (12), does not change the results for GBM and GMR.

Instead of an exogenously specified discount rate ρ one can make use of contingent claim analysis and assume that there exists a financial asset which is traded in a liquid market, and whose price dynamic is driven by the same noise generating process as the project value, in our case the one dimensional Brownian motion $W(t)$. Such an asset is generally referred to as a total spanning asset. It simplifies the analysis to assume that this asset has the same volatility structure, however this is not strictly necessary. We denote its drift rate with μ. In this case, Dixit and Pindyck argue that the optimal stopping problem

$$F(V) = \max_{\tau}\left(\mathbb{E}\left[(V(\tau) - I)\,\mathrm{e}^{-r\tau}\right]\middle|\, V(0) = V\right) \tag{13}$$

is equivalent to evaluating a perpetual American call on a hypothetical stock, paying a dividend rate of

$$\delta(V) = \mu - \frac{\alpha(V(t))}{V(t)}, \tag{14}$$

called the implied proportional dividend rate. Applying a Black–Scholes type analysis, they obtain the differential equation

$$\frac{1}{2}\sigma^2(V)F''(V) + (r - \delta(V))VF'(V) - rF(V) = 0 \tag{15}$$

for the value function $F(V)$. Eqs. (15) and (7) are of similar structure, r replaces ρ and $(r - \delta(V))$ replaces $\alpha(V)$. The advantage of (15) is that it does not include the ad hoc discount rate ρ, which in general is difficult to specify. The two conditions (8) and (9) remain the same. On the other side the contingent claims approach depends on the existence of a total spanning asset. This assumption is in some cases unrealistic. We refer to Henderson (2006) and Ewald and Yang (2008) for a utility based approach which operates in the presence of a partial spanning asset and includes risk aversion toward idiosyncratic risk. We do not however follow up with these ideas in the current article.