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Cost relationships in stochastic inventory systems: A simulation study of the $(S, S-1, t=1)$ model

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ABSTRACT

The analysis of the full stochastic model in which both the demand per unit time and the lead time are stochastic is complex. Analysis of the reduced stochastic inventory models in which only one of the parameters (either the demand per unit time or the lead time) is stochastic and the other is constant is relatively less complex.

In this paper we exploit insights from vector analysis and postulate an approximation that expresses the optimal cost of the full stochastic model in terms of the optimal costs of the two reduced models. We demonstrate the adequacy of the cost relationship in the context of one specific type of inventory model—a periodic review $(S, S-1, t=1)$ model—by performing an extensive set of simulations, using the Poisson, the exponential, and the gamma distributions to characterize demand and lead time. We also use the simulation data to develop regression relationships between the cost and an appropriate measure of variability, such as the standard deviation, the variance, or the coefficient of variation. For the cost of the full model, we find in our computations that our approximations have 98.4% accuracy for the Poisson, 96% for the exponential, and 97% for the gamma.

1. Introduction

Consider the following inventory models: the model where demand, D, is *iid* but lead time, L, is deterministic that we call the $(D,0)$ model; the one where D is deterministic but L is *iid* that we call the $(0,L)$ model; and the one where both D and L are *iid* that we call the (D,L) model. We wish to determine whether cost relationships exist between these models. But one of the complexities is that in dealing with the $(0,L)$ or (D,L) models, we must address the issue of order crossover. For when lead times are *iid*, orders may not arrive in the same sequence in which they were placed. A detailed description of inventory order crossovers and their types in practice is found in Riezebos (2006) and Riezebos and Gaalman (2009). In general, we can classify the different models based on whether D and L are deterministic or stochastic, and Table 1 shows the different models along with the expressions for the mean, $E(X)$, and the variance, $Var(X)$, of lead time demand (X). Please note that the variance of the demand during lead for the (D,L) model is additive in terms of those for the $(D,0)$ and the $(0,L)$ models.

In Table 1, *Model* (0,0) is the *EOQ* model with backlogging or planned shortages (Hax and Candea, 1984, pp. 136–138; Anderson et al., 2004, pp. 476–480); *Model* $(D,0)$ is the probabilistic construction in Hadley and Whitin (1963) or Silver et al. (1998); *Model* $(0,L)$ is the stochastic lead time/deterministic demand model in Liberatore (1979) and others, such as Sphicas (1982), Sphicas and Nasri (1984), and Kim et al. (2004). However, these models overstate the cost, because they do not account for order crossover. So in this paper, our primary concern rests with the total cost per unit time in the three different stochastic models in the above classification. A relationship that expresses $C(D,L)$ in terms of $C(D,0)$ and $C(0,L)$ would be a contribution to the inventory management literature.

Our goal in this paper is to draw upon the insights from vector analysis and postulate an approximate relationship that expresses the optimal cost of the full stochastic model (*i.e.*, one in which both lead time and demand are uncertain) in terms of the optimal costs of the two reduced models (*i.e.*, ones in which only the lead time or the demand is uncertain) and then to examine the adequacy of this relationship using simulation in the context of $(S-1, S)$ inventory systems with Poisson, exponential, and gamma lead times.

The rest of the paper is organized as follows. In Section 2, we present a literature review of the uses of the $(S-1,S)$ model, of recent developments in the study of order crossover, and of the

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Table 1
Our classification of stochastic inventory models with means and variances.

	Demand rate, *D*, deterministic	Demand rate, *D*, stochastic
Lead time, *L*, deterministic	*Model* (0 0): $E(X)=E(L).E(D)$. $Var(X)=0$	*Model* (*D*,0): $E(X)=E(L).E(D)$ $Var(X)=E(L).Var(D)$
Lead time, *L*, stochastic	*Model* (0,*L*): $E(X)=E(D).E(L)$ $Var(X)=E(D)^2.Var(L)$	*Model* (*D*,*L*): $E(X)=E(D).E(L)$ $Var(X)=E(L).Var(D)+[E(D)]^2.Var(L)$

literature on the (*D*,0), (0,*L*), and (*D*,*L*) inventory models. In Section 3, we present the conceptual underpinnings of the approximate cost relationship we propose and describe the inventory model for which we carry out the simulations. In Section 4, we present the results of the simulations and check the accuracy of the postulated cost relationship, which turns out to be 96% or above. In Section 5, we present regression relationships to provide insights on how costs relate to variability. Section 6 has the summary and conclusions. We attach an appendix for a list of symbols and acronyms used.

2. Literature review

Our present paper deals with how the costs for the models (*D*,0) and (0,*L*) can be combined in some fashion to produce the optimal cost for the full (*D*,*L*) model. And, because of its simplicity, we use as a vehicle the $(S, S-1)$ inventory model as a paradigm. However, for completeness, it behooves us to comment on previous uses of the $(S, S-1)$ model. Also, since the (0,*L*) model is subject to order crossover, we must review the recent literature on that and also on what has already been written on the (*D*,0), (0,*L*), and (*D*,*L*) models.

2.1. $(S-1,S)$ inventory systems

The $(S-1,S)$ inventory system is a special case of the order-point, order-up-to-level (*s*, *S*) system, with $s=S-1$, meaning that we place an order to bring the order-up-to-level to S with any occurrence of demand. This system has had applications in military spare-parts supply, the most famous of which has been METRIC (Sherbrooke, 1968), a FORTRAN IV computer model capable of determining base and depot stock levels for a group of recoverable items and designed for weapon-system levels, where a particular weapon may be demanded at several bases and the bases are supported by one central depot.

An $(S-1,S)$ policy means that an order is placed at each occurrence of a demand (Schmidt and Nahmias, 1985, p. 720). The $(S-1,S)$ model is appropriate when demand is low (*i.e.*, slow-moving items), but the item under consideration is expensive, so that the cost of ordering is negligible as compared to the cost of holding and shortages (Das, 1977, p. 836; Moinzadeh, 1989, p. 472). In a subsequent article concerned with emergency orders, Moinzadeh and Schmidt (1991) analyze a policy that allows up to *S* outstanding orders in the lost-sales case and an unlimited number of outstanding orders in the backlogging case (as opposed to models where there is no more than one outstanding order). And in a seminal article, Feeney and Sherbrooke (1966) derived steady state probabilities with demand compound Poisson for various state variables of the $(S-1, S)$ system and analyzed the backlogging and lost-sales cases for any distribution of replenishment time. A variant of the model is that of customer impatience when the system is out of stock and customers wait a certain time before canceling their orders (Das, 1977, p. 835).

Finally, the literature on $(S-1,S)$ systems is concerned with deriving expressions for steady-state measures of the operating characteristics of such systems in scenarios, for example, of Poisson demands and an arbitrary distribution of lead time (Smith, 1977) or constant resupply times and partial backorders (Moinzadeh, 1989). Rose (1972, p. 1020) best summarized the research up to that time:

The $(S-1, S)$ inventory model has been addressed … in different forms by several authors. Galliher et al. (1959) studied back-order cases where demand is arbitrary and delivery times constant, and where demand is Poisson distributed and delivery times exponentially distributed. Hadley and Whitin (1963) extended the model for both the back-order and noncaptive cases when demand is Poisson distributed and the distribution of delivery time is arbitrary. Feeney and Sherbrooke (1966) have … generalized the problem for the case of compound Poisson demand. Finally, Gross and Harris (1969) have studied the problem when demand is compound Poisson and the delivery time is related to the number of backorders.

2.2. Order crossover

A literature review of order crossover can be found in Robinson et al. (2001) and Bradley and Robinson (2005). Also, a comprehensive literature review can be found in Hayya et al. (2008), and we can trace this type of research to Finch (1961), Agin (1966), and Zalkind (1976, 1978). So if we take the following equation from Zalkind (1976, Lemma 3.3):

$$Var(ELT)=\int_0^{\infty} F_l(1-F_l)dl, \tag{2.1}$$

where F_L is the *cdf* of the parent lead time, we can develop that into (Hayya, et al., 2008)

$$Var(ELT)=E(L)-E(L^{1,2}), \tag{2.2}$$

where *ELT* is the effective lead time (a transformation of the original lead time, *L*, after order crossover) and $E(L^{1,2})$ denotes the first order statistic from a random sample of size 2 from the distribution of *L*. We define the effective lead time (*ELT*) as the time interval between the *ith* order placement and the *ith* order arrival, with the index *i* not identified (or tagged) to any particular order. Please note that Eq. (2.2) does not imply at most two orders outstanding; it is just how Eq. (2.1) works out, and the number of orders outstanding can be 1, 2, 3, … However, the ordering interval is one.

For an ordering interval *T* equal to one, *Var*(*ELT*) would be bracketed, as

$$E(L)-E(L^{1,2})\le Var(ELT)\le Var(L). \tag{2.3}$$

For example, for an ordering interval *T* equal to one, if $L\sim \text{Exp}(\lambda)$ whose mean and standard deviation are both equal to $(1/\lambda)$, we have

$$\frac{1}{2\lambda}\le Var(ELT)\le \frac{1}{\lambda^2}. \tag{2.4}$$

We may consider Eq. (2.2) as the variance of the effective lead time in the long run, provided we order every period. In Table 2, we illustrate the development of *ELT* through simulation for the case where the lead times are exponentially distributed with mean 5. We see that, in the simulation covering a set of 10,000 consecutive orders, the variance of the effective lead times is equal to 2.41. This is within 3.6% of the theoretical value we get from Eq. (2.2). If $L\sim \text{Exp}(\lambda)$, we have

$$E(L^{1:2})=1/2\lambda,$$