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Mean-variance portfolio selection with only risky assets under regime switching

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ABSTRACT

This paper explores a portfolio selection model of multiple risky assets with regime switching. There are $n + 1$ risky assets in the financial market available to the mean-variance investors. The feasibility issue is solved by constructing an equivalent condition. We derive the analytical expressions of the efficient frontier and efficient feedback portfolio via three systems of ordinary differential equations that admit unique solutions. The mutual fund theorem is also proved. Several numerical examples are provided to demonstrate how the efficient frontier is affected by the market regime movement and the investor's time horizon.

1. Introduction

Portfolio selection is concerned with the allocation of the investor's assets amongst different types of financial securities so as to optimize the total return of the portfolio. Along with a desirable investment return, however, investors are also seeking to control the future uncertainties of their portfolio. Thus, a measure needs to be defined as a quantifiable indicator of the portfolio risk. Markowitz (1952) firstly gave an accurate definition of investment risk by applying the mathematical terminology of "variance" in the probability theory. Considering the trade-off between the mean and variance of a portfolio, an optimal investment strategy was achieved, known as "efficient portfolio". In a single period setting, nevertheless, Markowitz's mean-variance model failed to capture the dynamic process of portfolio selection facing investors in the real world. A number of literatures have been devoted to the extension of the original single period model to the multi-period case. For more details, the reader is referred to Pliska (1997) and Li and Ng (2000).

Since continuous-time finance theory was pioneered by Robert C. Merton in the 1970s, financial modelling in continuous-time setting has been thriving and employed to deal with a range of theoretical and practical problems. The continuous-time mean-variance portfolio selection model was originally formulated and solved by Zhou and Li (2000), which obtained both the efficient portfolio and efficient frontier in closed form by applying the stochastic linear-quadratic (LQ) control theory. Thereafter, their model has been extensively studied by numerous literatures. Lim and Zhou (2002) considered a complete market with bounded random coefficients in a general framework and obtained the efficient frontier by solving two backward stochastic differential equations. Chiu and Wong (2011) applied their technique to solve a mean-variance portfolio selection problem with cointegrated risky assets. The constant elasticity of variance (CEV) model was employed to characterize the evolution of a risky asset price in Shen et al. (2014). No bankruptcy constraint was explored in Bielecki et al. (2005) by using the martingale approach. Besides, a few papers also took market frictions into account. Li et al. (2002) imposed shorting prohibition on the trading of stocks while borrowing from the bank account was still permitted. Fu et al. (2010) supposed a spread between the interest rates for lending and borrowing. Furthermore, asset-liability management was studied under the mean-variance framework. Chiu and Li (2006) considered a dynamic liability process driven by a geometric Brownian motion. Xie et al. (2008) modelled an uncontrollable liability with a drifted Brownian motion. Leippold et al. (2011) introduced endogenous liabilities and obtained efficient portfolio and efficient frontier in a multi-period setting. Their model was paralleled to a continuous-time asset-liability management problem by Yao et al. (2013). Both the CEV process and geometric Brownian motions were used by Zhang and Chen (2016) to model the multiple asset processes and exogenous liability respectively in a complete market.

To better capture the random environment of the financial market, regime-switching models have been applied to some of the key financial parameters, such as interest rate, equity risk premium and stock volatility. The basic idea is that these financial parameters are supposed to move along with the underlying market state. For example, investors would anticipate a higher appreciation rate and a lower volatility when the stock market is believed to be bullish. In the previous literatures,

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the market regime is usually characterized by a Markov chain the value of which switches within a finite state space. Numerous models have been developed to solve some of the fundamental financial problems, such as asset pricing. See Buffington and Elliott (2002); Guo (2001) and Elliott et al. (2005). Analytical solutions were derived for a general investment-consumption model with regime switching in Sotomayor and Cadenillas (2009). The associated value function was solved explicitly with different types of consumption utility functions. A regime-switching model was originally formulated to solve the mean-variance portfolio selection problem by Zhou and Yin (2003). They obtained the explicit expressions of the efficient portfolio and efficient frontier via the solutions of two systems of linear ordinary differential equations (ODEs). Chen et al. (2008) extended their work by introducing a Markov-modulated geometric Brownian motion to model the insurance company's uncontrollable liability process. Similarly, the investor's exogenous liability was assumed to be a Markov-modulated Brownian motion in Xie (2009).

In this paper, we follow the work of Yao et al. (2014) as the first attempt to address a financial market without risk free assets. This hypothesis could reflect the stochastic nature of the interest rate over a long time horizon. Moreover, Markowitz (1952) proposed the mean-variance principle with the purpose of addressing the diversification problem of various stocks. The efficient frontier and global minimum variance were derived with the absence of risk free assets. Along this line, Yao et al. (2014) formulated a dynamic portfolio selection problem of only risky assets as a direct extension of Markowitz's single period model. In their paper, a different conclusion has been drawn regarding the efficient frontiers. In contrast with the static case, the capital market line in the continuous time model is strictly above the efficient frontier of a hyperbolic shape that corresponds to the case of only risky assets. This is due to the fact that investors continuously adjust its allocation to risk free assets to maintain an optimal strategy.

Our paper extends Yao et al. (2014) by considering a regime-switching financial market where all relevant parameters are driven by a continuous time Markov chain. We employ the Lagrange multiplier and “completion of square” technique in the LQ stochastic control theory. This commonly used approach is well applied because the investor's wealth process, although in a more general form, is still governed by a linear stochastic differential equation (SDE). By solving three systems of linear ODEs, we derive the efficient frontier and efficient feedback portfolio in closed form. Unsurprisingly, the efficient frontier is no longer a straight line, and the global minimum variance is strictly greater than zero, since there is no risk-free portfolio, that is, the investor cannot construct a dynamic portfolio so as to achieve a pre-specified investment return with zero variance at the terminal time.

The remaining part of the paper is outlined as follows. Section 2 formulates a continuous time mean-variance portfolio selection model of only risky assets under regime switching. One equivalent condition is proved for the problem feasibility, and the Lagrange multiplier is introduced in Section 3. In Section 4, the unconstrained dual problem is analytically solved via three systems of linear ODEs. Section 5 derives the efficient feedback portfolio, efficient frontier, global minimum variance and mutual fund theorem. Several numerical examples are provided to illustrate our results in Section 6. Section 7 gives a brief conclusion.

2. Problem formulation

Throughout the paper, let $(\Omega, \mathcal{F}, P)$ be a complete probability space, on which are defined an m-dimensional standard Brownian motion $W(t):=(W_1(t), \ldots, W_m(t))'$ and a continuous time stationary Markov chain $\alpha(t)$ with a finite state space $\mathcal{M} = \{1, 2, \ldots, l\}$ and a generator matrix $Q = (q_{ij})_{l\times l}$. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be the filtration generated by $W(t)$ and $\alpha(t)$ augmented by the null sets contained in $\mathcal{F}$. We assume the independence of $W(t)$ and $\alpha(t)$ to ensure that $W(t)$ is a standard Brownian motion with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. All the vectors are supposed to be column vectors. The transpose of any matrix A is denoted by A'. The norm $\|\cdot\|$ is defined as $\left\|A\right\| = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2}$, where $A = (a_{ij})_{m\times n}$.

We consider a financial market composed of $n + 1$ risky assets price processes of which, denoted by $S_i(\cdot)$, $i = 0, 1, 2, \ldots, n$, are characterized by the following Markov-modulated geometric Brownian motions

$$\begin{cases} dS_i(t) = S_i(t)\left[b_i(t, \alpha(t))dt + \sum_{j=1}^{m}\sigma_{ij}(t, \alpha(t))dW_j(t)\right], \\ S_i(0) = S_i^0, \end{cases}$$

where $b_i(t, k)$ and $\sigma_i(t, k):=(\sigma_{i1}(t, k), \ldots, \sigma_{im}(t, k))'$, $k = 1, 2, \ldots, l$, are the appreciation rate and volatility vector of the ith risky asset respectively, corresponding to the market regime k.

Suppose that an agent, with an initial wealth x_0, is investing in the $n + 1$ assets and adjusting his portfolio weights continuously within a finite time horizon $T > 0$. Both long and short positions are permitted without any transaction cost. The agent's wealth process $x(\cdot)$ would evolve as a linear SDE

$$\begin{cases} dx(t) = [b_0(t, \alpha(t))x(t) + B(t, \alpha(t))'u(t)]dt \\ \qquad\qquad + [x(t)\sigma_0(t, \alpha(t))' + u(t)'\sigma(t, \alpha(t))]dW(t), \\ x(0) = x_0, \quad \alpha(0) = i_0, \end{cases} \tag{2.1}$$

where i_0 is the initial market mode and $u(\cdot):=(u_1(\cdot), \ldots, u_n(\cdot))'$ is defined as the agent's portfolio vector, the kth element of which represents the market value of the kth risky asset held by the agent. The remaining part $x(t) - \sum_{k=1}^{n} u_k(t)$ is allocated to the 0th asset. Both $B(t, \alpha(t))$ and $\sigma(t, \alpha(t))$ are defined as below

$$B(t, \alpha(t)):=[b_1(t, \alpha(t)) - b_0(t, \alpha(t)), \ldots, b_n(t, \alpha(t)) - b_0(t, \alpha(t))]', \sigma(t, \alpha(t))$$
$$:=[\sigma_1(t, \alpha(t)) - \sigma_0(t, \alpha(t)), \ldots, \sigma_n(t, \alpha(t)) - \sigma_0(t, \alpha(t))]'.$$

Note that $\sigma(t, \alpha(t))$ is a matrix of order $n \times m$.

Remark 2.1. If $\sigma_0(t, \alpha(t)) \equiv 0$, the 0th risky asset could be taken as a risk free bank account which yields a predictable future return regardless of the market randomness modelled by the Brownian motion $W(\cdot)$. In this particular scenario, the agent's wealth process would reduce to (2.6) in Zhou and Yin (2003).

Before we formulate the mean-variance portfolio optimization problem, several assumptions need to be made for technical convenience.

Assumption 2.1. $b_i(\cdot, k)$ and $\sigma_{ij}(\cdot, k)$ are Borel-measurable and bounded functions of t for $i = 0, \ldots, n$, $j = 1, \ldots, m$, $k = 1, \ldots, l$.

Assumption 2.2. $\sigma(t, i)$ satisfies the nondegeneracy condition, i.e., there exists $\delta > 0$ such that $\Sigma(t, i):=\sigma(t, i)\sigma(t, i)' \geq \delta I$, $\forall\, t \in [0, T]$, $i = 1, \ldots, l$, where I denotes the n-dimensional identity matrix.

Remark 2.2. The nondegeneracy condition in Assumption 2.2 could be satisfied only if the rank of $\sigma(t, i)$ is n, which implies that the dimension of $W(\cdot)$ must be at lease equal to the number of risky assets in the financial market. However, the market completeness is unnecessary, that is, m may be strictly greater than n.

Definition 2.1. A portfolio $u(\cdot)$ is said to be admissible if it is an $\mathcal{F}_t$-adapted locally integrable process, i.e., $\int_0^T \| u(t)\|^2\, dt < \infty$, a.s. and the SDE (2.1) admits a unique strong solution $x(\cdot)$ satisfying the square-integrable condition, i.e, $E\max_{0\leq t\leq T}x(t)^2 < \infty$. Let U denote the set of all admissible portfolios.

Remark 2.3. Due to its linear structure, the wealth process (2.1) always has an explicit solution for any locally integrable process $u(\cdot)$. By Definition 2.1, therefore, the essential difficulty is to show the integrability of $\max_{0\leq t\leq T}x(t)^2$ when verifying the admissibility of a portfolio process.

As a mean-variance investor, the agent's objective is to find an