ISSN 0005-1179, Automation and Remote Control, 2015, Vol. 76, No. 6, pp. 957–976. © Pleiades Publishing, Ltd., 2015.
Original Russian Text © B.T. Polyak, A.A. Tremba, M.V. Khlebnikov, P.S. Shcherbakov, G.V. Smirnov, 2015, published in Avtomatika i Telemekhanika, 2015, No. 6, pp. 18–41.

Large Deviations in Linear Control Systems with Nonzero Initial Conditions

B. T. Polyak[*,], A. A. Tremba[*], M. V. Khlebnikov[*], P. S. Shcherbakov[*], and G. V. Smirnov[***]**

[*] *Trapeznikov Institute of Control Sciences, Russian Academy of Science, Moscow, Russia*
[**] *ITMO University, St. Petertsburg, Russia*
[***] *University of Minho, Braga, Portugal*
e-mail: boris@ipu.ru, atremba@ipu.ru, khlebnik@ipu.ru, sherba@ipu.ru, smirnov@math.uminho.pt

Received August 5, 2014

Abstract—Research in the transient response in linear systems with nonzero initial conditions was initiated by A.A. Feldbaum in his pioneering work [1] as early as in 1948. However later, studies in this direction have faded down, and since then, the notion of transient process basically means the response of the system with zero initial conditions to the unit step input. A breakthrough in this direction is associated with the paper [2] by R.N. Izmailov, where large deviations of the trajectories from the origin were shown to be unavoidable if the poles of the closed-loop system are shifted far to the left in the complex plane.

In this paper we continue the analysis of this phenomenon for systems with nonzero initial conditions. Namely, we propose a more accurate estimate of the magnitude of the peak and show that the effect of large deviations may be observed for different root locations. We also present an upper bound on deviations by using the linear matrix inequality (LMI) technique. This same approach is then applied to the design of a stabilizing linear feedback aimed at diminishing deviations in the closed-loop system. Related problems are also discussed, e.g., such as analysis of the transient response of systems with zero initial conditions and exogenous disturbances in the form of either unit step function or harmonic signal.

DOI: 10.1134/S0005117915060028

1. INTRODUCTION

Consider the stable linear system

$$\dot{x} = Ax, \quad x(t) \in \mathbb{R}^n, \tag{1}$$

with *nonzero initial condition* $x(0)$. It is of great interest to analyze the transient of the system, i.e. the behavior of its trajectory $x(t)$ for all $t \geqslant 0$.[1] In particular, it is highly desirable to have estimates of the quantity

$$\xi(x(0)) = \max_{t\geqslant 0} \frac{|x(t)|}{|x(0)|}$$

(here, $|\cdot|$ is a vector norm), i.e., the maximal deviation of the trajectory from the origin during the transient. If this quantity is large, we say that *large deviations* take place; in case these deviations are observed at the initial part of the trajectory, we refer to these as the *peak effect.* Needless to say, such characteristics of the transient state are among the most important ones, having transparent engineering interpretations.

[1] Clearly, by stability we have $\lim_{t\to\infty} x(t) = 0$.

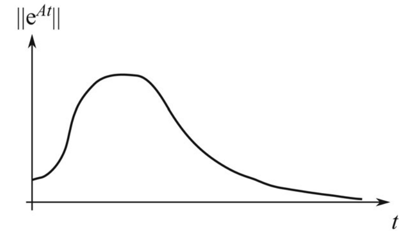

Fig. 1. Typical behavior of the $\| \mathrm{e}^{At} \|$ function.

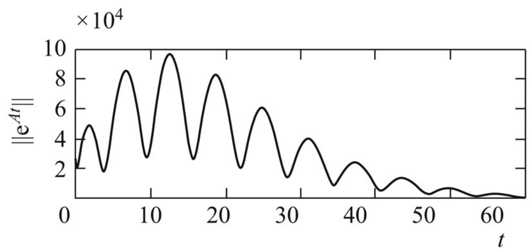

Fig. 2. Plot of the matrix exponential in the stabilization problem for Boeing 767.

Notably, nonzero initial conditions appear quite naturally in various situations, for instance in stabilization and control by means of observers [3] with unknown initial conditions; this is also typical of switching systems [4] that reach a certain point in the phase space by a switch time instant. In Section 5 we show that the nature of peak effects (deviations) is very close to that of the well-studied phenomenon of *overshoot* in systems with zero initial conditions and unit step input.

It is also worth mentioning that deviations of the solutions of linear ordinary differential equations with nonzero initial conditions are the subject of interest not only in control theory but also in the computational mathematics. Since the solution of system (1) has the form

$$x(t) = \mathrm{e}^{At} x(0),$$

we have

$$\xi \doteq \max_{t \geqslant 0} \max_{|x(0)|=1} \frac{|x(t)|}{|x(0)|} = \max_{t \geqslant 0} \| \mathrm{e}^{At} \|,$$

where $\| \cdot \|$ is the corresponding induced matrix norm, so that evaluation of the magnitude of deviation is directly related to the estimation of the *matrix exponential*.

In the celebrated paper [5] (also see its continuation [6]) on the methods of computing the matrix exponential, the existence of a noticeable maximum (often quite significant) of the function $\| \mathrm{e}^{At} \|$ is considered typical (it is referred to as *hump* in the above mentioned works); see Fig. 1 borrowed from [6].

Several examples illustrating the effects of very large deviations in "real-world" stabilization problems for control systems are also given in [6]; for instance, in the 55-dimensional stabilization problem for Boeing 767, the maximum of the quantity $\| \mathrm{e}^{At} \|$ was of the order of 10^5, see Fig. 2. However, in these works, no attention has been paid to the *evaluation* of the magnitude of the hump.