Journal of Economic Dynamics & Control 76 (2017) 136–151

Contents lists available at ScienceDirect

Journal of Economic Dynamics & Control

journal homepage: www.elsevier.com/locate/jedc

Equal risk pricing under convex trading constraints

Ivan Guo*, Song-Ping Zhu

School of Mathematics and Applied Statistics, University of Wollongong, NSW 2522, Australia

ARTICLE INFO

Article history:
Received 19 June 2016
Revised 12 January 2017
Accepted 16 January 2017
Available online 23 January 2017

MSC:
91G10
91G20

Keywords:
Derivative pricing
Trading constraints
Equal risk price
Short-selling ban

ABSTRACT

In an incomplete market model where convex trading constraints are imposed upon the underlying assets, it is no longer possible to obtain unique arbitrage-free prices for derivatives using standard replication arguments. Most existing derivative pricing approaches involve the selection of a suitable martingale measure or the optimisation of utility functions as well as risk measures from the perspective of a single trader.

We propose a new and effective derivative pricing method, referred to as the *equal risk pricing approach*, for markets with convex trading constraints. The approach analyses the risk exposure of both the buyer and seller of the derivative, and seeks an *equal risk price* which evenly distributes the expected loss for both parties under optimal hedging. The existence and uniqueness of the equal risk price are established for both European and American options. Furthermore, if the trading constraints are removed, the equal risk price agrees with the standard arbitrage-free price.

Finally, the equal risk pricing approach is applied to a constrained Black–Scholes market model where short-selling is banned. In particular, simple pricing formulas are derived for European calls, European puts and American puts.

1. Introduction

In 2008, regulators from several countries enforced various levels of naked short-selling restrictions in response to the financial crisis. Financially, it is believed that the banning of short-selling would reduce the volatility and weaken the negative impacts of a downturn market. Mathematically, on the other hand, it poses a significant challenge in the modelling and pricing of derivatives. Specifically, how much do the prices of derivatives change if the short-selling of the underlying asset is banned? The absence of short-selling is an example of a *convex trading constraint*. The existence of trading constraints invalidates the fundamental assumption of perfect liquidity in the classical Black–Scholes model, resulting in an incomplete market. In this scenario, it is not possible to obtain unique arbitrage-free prices for all options via replication arguments, and the uniqueness of the martingale measure is not guaranteed. Super- and sub-replication arguments can be used to produce upper and lower pricing bounds. These bounds correspond to prices at which either the seller or buyer can hedge without risk.

The problem of derivative pricing in incomplete markets has been studied extensively in the literature, with a large number of approaches and techniques. Amongst these approaches, there are several recurring themes. These include: martingale measure selections, utility maximisation and risk measures. In the first category, there have been numerous attempts to choose an appropriate equivalent martingale measure as the pricing measure according to some optimal criterion, such as the minimal martingale measure by Föllmer and Schweizer (1991), the minimal entropy martingale measure by Frittelli (2000), the minimax measure by Bellini and Frittelli (2002) and the minimal distance martingale measure by

* Corresponding author.
E-mail address: ivanguo1986@gmail.com (I. Guo).

http://dx.doi.org/10.1016/j.jedc.2017.01.005
0165-1889/

Goll and Rüschendorf (2001). In these works, existence results of the respective measures are provided and often they are closely related to utility maximisation problems. In practice, the martingale measure is interpreted to be implicitly chosen by the market and is computed via calibration.

Utility based approaches usually involve first choosing a utility function for the trader, then the derivative is priced so that the utility is unaffected if a marginal amount of the derivative is included in an optimal trading strategy. Examples in this category include Karatzas and Kou (1996), Davis (1997), Rouge and El Karoui (2000), Musiela and Zariphopoulou (2004) and Hugonnier et al. (2005). In another variation, the acceptability of the derivative price is determined by evaluating a trader's utility or risk exposure. One such example is the 'indifference price' from Henderson and Hobson (2009). Elsewhere, Artzner et al. (1999) proposed the coherent risk measures, which were then generalised into convex risk measures by Föllmer and Schied (2002). Xu (2006) applied convex risk measures to incomplete markets and introduced the notion of risk efficiency. Xu (2006) then showed that the expected shortfall measure, the Value-at-Risk approach, as well as the pricing measures studied in Carr et al. (2001) are all contained in the same framework.

The utility and risk measure based approaches mentioned above have a common feature: they only produce the maximal or minimal *acceptable price* for a single trader, but not an explicit *transactional price*. If the derivative is deemed acceptable for both the buyer and seller, the two resulting acceptable prices once again only provide a price interval or spread. In order to more precisely determine the changes in derivative prices due to trading constraints, as well as modelling the price negotiation, we require an approach which directly models the transactional price within the interval.

To this end, we propose a completely new approach, referred to as the *equal risk pricing approach*, which determines the derivative price by simultaneously analysing the risk exposure of both parties involved in the contract. The idea of considering the risk exposure of both parties was previously explored in Barrieu and El Karoui (2004, 2005), albeit for a different problem. The focus in Barrieu and El Karoui (2004, 2005) was to optimally design a derivative security that is mutually acceptable, while allowing for maximal risk transfer from one agent to another under a convex risk measure. In this paper, we aim to find a fair price for a fixed derivative while distributing risk evenly between the two parties. The resulting price is referred to as the *equal risk price*. It can be interpreted as a fair price that both parties are happy to accept during negotiation if they intend to enter into the derivative contract. It is worth stressing that the equal risk price is not simply a variation of the existing approaches such as the indifference price Henderson and Hobson (2009), with differences in both interpretation and application (see Remark 3.2).

Since this is the first paper illustrating the equal risk pricing approach, we will focus on a simple model which is mathematically tractable and has clear financial interpretations. Specifically, we examine the case where a convex trading constraint is imposed on an otherwise complete and arbitrage-free market. Individual risk functions are used to model the risk exposure of the buyer and seller. The existence and uniqueness of the equal risk price are established for arbitrary European and American options. Furthermore, our model is consistent with standard arbitrage-free pricing models if the market is complete, since all equal risk pricing results agree with their arbitrage-free pricing counterparts upon the removal of trading constraints. Finally, in the case of short-selling being banned in the Black–Scholes market model, simple pricing formulas are produced for common options such as the European call, European put and American put.

The paper is organised as follows. Section 2 specifies the underlying model and the convex trading constraints. Sections 3 and 4 address European and American options, respectively. In particular, Sections 3.1 and 4.1 establish the existence and uniqueness of the equal risk prices, with Theorems 3.1 and 4.1 being the main results. Then Sections 3.2 and 4.2 show that the prices are consistent with the standard arbitrage-free prices if the trading constraints were removed. Sections 3.3 and 4.3 consider the scenario where short-selling is banned in the Black–Scholes model. Theorem 3.2 provides a simple and elegant pricing formula for European options with monotonic payoffs, which can be directly applied to calls and puts. Finally, Theorem 4.2 provides an analogous pricing formula for the American put in terms of American put prices under the standard, unrestricted Black–Scholes model.

2. Model specification

We begin by presenting our general market model. Let $(\Omega, \mathbb{F}, \mathbb{Q})$ be a probability space where the filtration $\mathbb{F} = \{\mathcal{F}_t : t > 0\}$ represents the information flow available to market participants. We assume that there exists at least one risky asset available for trading and the discounted prices of all tradable assets are $\mathbb{F}$-progressively measurable processes. Furthermore, the market containing all self-financing, admissible and progressively measurable trading strategies is assumed to be arbitrage-free. Hence there exists at least one equivalent martingale measure $\mathbb{Q}$. In particular, the dynamics of all discounted underlying assets are $\mathbb{Q}$-martingales.

The market also contains trading constraints. In order to describe the set of permissible trading strategies, we shall focus our attention on the set of *gains* processes, or self-financing portfolios which require zero initial capital. This is sufficient for our purposes since every wealth process can be decomposed into its initial value and its gains process. In the current study, we also require the set of permissible wealth processes to be *convex*. Explicitly, this restriction is stated in the following definition.

Definition 2.1 (Gains process, Convex trading constraints)**.** For each $\mathbb{Q}$-admissible, self-financing trading strategies available in the market that requires zero initial capital, we call its discounted wealth process a *gains process*. Denote by $\mathbb{G}$ the set of all possible gains processes in the market. The set $\mathbb{G}$ is assumed to be convex and to contain the zero process.