International Journal of Production Economics 200 (2018) 130–133

Contents lists available at ScienceDirect

International Journal of Production Economics

journal homepage: www.elsevier.com/locate/ijpe

Technical note: Solving inventory models by algebraic method

Xu-Ren Luo [a], Chih-sheng Chou [b,*]

[a] Department of Computer Science and Information Engineering, Chung Cheng Institute of Technology, National Defense University, Taiwan, ROC
[b] Data Analytics, Sr., The MITRE Corporation, McLean, VA 22102, USA

ARTICLE INFO

Keywords:
Inventory model
Economic order
Production quantity
Shortage

ABSTRACT

We consider the open question raised by Chang et al. (2005) to solve the EOQ and EPQ inventory models without referring to calculus. Lau et al. (2016) and Chiu et al. (2017) both extended this open question by deriving criterion for the existence and uniqueness of the interior minimum solution but they used analytical techniques that are related to calculus. Moreover, their derivations are incomplete and contained questionable results. In this note, we only used algebraic approach for their extended open question.

1. Introduction

Since Grubbström and Erdem (1999) used an algebraic approach to solve the minimum problem of an inventory model, there are nearly two hundred papers that followed this trend to develop different algebraic methods to find the optimal solutions for inventory systems. Most papers concentrated on their own inventory models and did not pay attention to an open question proposed by Chang et al. (2005), which contains a quadratic polynomial inside a square root and requires finding the optimal solution only by algebraic methods without referring to calculus. Until recently, Lau et al. (2016) reconsidered the open question and extended to a more general setting. They tried to obtain criterion to guarantee the existence and uniqueness of the optimal solution for the minimum problem. Chiu et al. (2017) pointed out the findings of Lau et al. (2016) contained questionable results with improper analytic approach and inadequate partitioning of the solution space, and provided an improvement. However, we find that the improvement of Chiu et al. (2017) is right but incomplete. Moreover, they used calculus method to obtain their results. In Chang et al. (2005), the original restriction is to derive the optimal solution by algebraic approach without referring to calculus.

The purpose of this note is to present a further revision of Chiu et al. (2017) with algebraic approach and answer the open question proposed by Chang et al. (2005) as a corollary. A major contribution of this technical note is our proposed derivation method that will be useful for researchers and practitioners to solve the minimum problems under algebraic approaches. In Section 2, a brief review of merit achievements associated with the inventory modeling for the past decade is presented. It is followed by our proposed methodology in Section 3, and numerical applications in Section 4. Finally, a synthesis discussion in Section 5 will conclude our technical note.

2. Brief review for inventory modeling achievements

To be compatible with Càrdenas-Barrón (2001), Chang et al. (2005), Lau et al. (2016) and Chiu et al. (2017), we use the same notation and expressions as theirs. Readers for this note are suggested referring to them for notation and assumptions.

The original inventory model solved by algebraic method was proposed by Càrdenas-Barrón (2001), and further revised by Ronald et al. (2004) and Chang et al. (2005) as

$$C(Q,B) = \frac{b+h}{2\rho\, Q}B^2 - hB + \frac{h\rho}{2}Q + \frac{KD}{Q} + cD. \tag{1}$$

Specifically, Chang et al. (2005) mentioned that an alternative way to solve their minimum problem is to solve the following problem:

$$C(Q(B),B) = cD + h\left(\sqrt{\left(1+\left(\frac{b}{h}\right)\right)B^2 + \frac{2\rho}{h}KD} - B\right). \tag{2}$$

By Equation (2), they provided the next open minimum problem:

$$\sqrt{(1+\alpha)B^2+\beta} - B \tag{3}$$

without using partial derivatives of calculus, where $\alpha = b/h$, $\beta = 2\rho KD/h$, $\alpha > 0$ and $\beta > 0$.

Recently, there are two papers: Lau et al. (2016) and Chiu et al. (2017) to consider the following more generalized minimum problem:

* Corresponding author.
E-mail addresses: nrgman.luo@gmail.com (X.-R. Luo), chihsheng@mitre.org (C.-s. Chou).

https://doi.org/10.1016/j.ijpe.2018.03.023
Received 6 April 2017; Received in revised form 14 October 2017; Accepted 25 March 2018
Available online 27 March 2018
0925-5273/

$$f(x) = \sqrt{ax^2 + bx + c} - x \tag{4}$$

with $f(x) > 0$, for $x > 0$, to secure the minimum problem has an interior optimal solution.

Lau et al. (2016) obtained two cases:

(a) $a > 1$, $b \le 0$, $c > 0$ and $4ac > b^2$, and
(b) $a > 1$, $b > 0$, $c > 0$ and $4c > b^2$.

However, their solution contained questionable results, which will be demonstrated by our Theorem 1 in Section 3, and several of their derivations were derived by calculus.

Chiu et al. (2017) claimed two cases:

(a) $a > 1$, $c > 0$ and $4ac > b^2$, and
(b) $a > 1$ and $4c > b^2$.

They used the knowledge of calculus in derivations and their findings are incomplete, which will be demonstrated by our Theorem 1 as well. Hence, in this note, we will provide further improvements for Lau et al. (2016) and Chiu et al. (2017) with algebraic method.

3. Proposed algebraic method

Our goal is to find conditions to guarantee that $f(x) = \sqrt{ax^2 + bx + c} - x$ for $x > 0$ with $f(x) > 0$ has a unique minimum (optimal) solution by algebraic methods. Before proving our theorem, we would firstly explain the reason why we concentrate on $x > 0$ with $f(x) > 0$ herein.

When $x \to 0^+$ and $f(x) \to 0$, it implies that the inferior value occurs on the boundary. Thus, the original inventory model has an inferior value when $Q \to 0^+$, which is violating the common sense of the original minimum cost inventory model. As a Remark, for the original minimum cost inventory model, the average set up cost will go to infinite when the replenishment cycle approaches to zero. Therefore, the original minimum cost inventory model cannot have the finding as $Q^* \to 0^+$. If the optimal solution of $f(x)$ satisfies $x^* \to \infty$, it implies that the optimal order quantity as well as the holding cost will go to infinite. Thus, $x^* \to \infty$ is not an acceptable optimal solution for $f(x)$. Under the condition of a point, say $\hat{x}$, satisfying $0 < \hat{x} < \infty$ and $f(\hat{x}) \le 0$, the corresponding inventory model will have negative or zero holding cost and shortage cost that is a contradiction for inventory models. Hence, we look for restrictions to guarantee $x > 0$ and $f(x) > 0$.

Remark. We want to rule out the case of $a = 1$, $c \ge 0$ and $b = 2\sqrt{c}$ to result in the trivial case $f(x) = \sqrt{x^2 + 2\sqrt{c}x + c} - x \equiv \sqrt{c}$ that has infinite minimum solutions.

In this section, we will prove two necessary conditions of $c > 0$ and $a > 1$, and rewrite Equation (4) to obtain additional three conditions: $4ac - b^2 > 0$, $4c > b^2$ when $b \ge 0$, and $4(a-1)c > b^2$ when $b < 0$ for deriving our proposed Theorem 1.

First, we will prove that $c > 0$. Assuming $c < 0$, we select a point x_1 with $x_1 > 0$ and $ax_1^2 + bx_1 + c < 0$, which is a contradiction with $\sqrt{ax_1^2 + bx_1 + c}$.

Thus, we select $x_1 = \frac{1+|b|}{1+|a|} s$ with $s = \min\left\{1, \frac{|c|(1+|a|)}{2(1+|b|)^2}\right\}$. As $s \le 1$, we compute

$$\begin{aligned} ax_1^2 + bx_1 + c &< (1+|a|)x_1^2 + (1+|b|)x_1 - |c| = \frac{(1+|b|)^2}{1+|a|}(s^2 + s) - |c| \\ &\le 2s\frac{(1+|b|)^2}{1+|a|} - |c| \le 0 \end{aligned} \tag{5}$$

to imply an unacceptable result $f(x_1) < 0$.

With an assumption of $c = 0$, we construct a sequence (t_n) as $t_n = \frac{1}{n^2}$ and $t_n \le \sqrt{t_n}$, and evaluate that

$$\begin{aligned} f(t_n) &= \sqrt{at_n^2 + bt_n} - t_n \le \sqrt{|a|t_n^2} + \sqrt{|b|t_n} + t_n \le (1 + |a| + |b|)\sqrt{t_n} \\ &= \frac{1 + |a| + |b|}{n} \end{aligned} \tag{6}$$

to yield $f(t_n) \to 0$. As it will result in a solution of $x^* \to 0^+$, we derive the first condition of

$$c > 0. \tag{7}$$

Second, we will prove that $a > 1$. When $a < 1$, we take $x_0 = m\left(\frac{1+|b|}{1-a}\right)$ with $m = 1 + \frac{\sqrt{(1-a)|c|}}{1+|b|} > 1$ to imply that

$$\begin{aligned} (1-a)x_0^2 - (1+|b|)x_0 &= (1-a)m^2\left(\frac{1+|b|}{1-a}\right)^2 - (1+|b|)m\left(\frac{1+|b|}{1-a}\right) \\ &= \frac{(1+|b|)^2}{1-a}m(m-1) > \frac{(1+|b|)^2}{1-a}(m-1)^2 \\ &> \frac{(1+|b|)^2}{1-a}\frac{(1-a)|c|}{(1+|b|)^2} = |c| \ge c. \end{aligned} \tag{8}$$

From $(1-a)x_0^2 > (1+|b|)x_0 + c > bx_0 + c$, it yields $f(x_0) < 0$ that is a contradiction, and thus, $a < 1$ is not acceptable. The condition of $a \ge 1$ is derived. We further show that when $a = 1$, a minimum solution for $0 < x < \infty$ cannot be found. Note that the inferior value occurs on the boundary $x = 0$ or $x = \infty$ are out of the domain $0 < x < \infty$. When $a = 1$, three cases are elaborated separately herein: (a) $b > 2\sqrt{c}$, (b) $b < 2\sqrt{c}$ and (c) $b = 2\sqrt{c}$.

For case (a), when $a = 1$ and $b > 2\sqrt{c}$, we derive that

$$\begin{aligned} 4c < b^2 &\Leftrightarrow ax^2 + bx + c > x^2 + 2\sqrt{c}x + c \Leftrightarrow \sqrt{ax^2 + bx + c} > x + \sqrt{c} \Leftrightarrow f(x) \\ &> \sqrt{c}. \end{aligned} \tag{9}$$

Thus, the inferior value occurs when $x \to 0^+$.

For case (b), $a = 1$ and $b < 2\sqrt{c}$, we show that $f(x) > b/2$ for all $x > 0$ in

$$\begin{aligned} f(x) > b/2 &\Leftrightarrow \frac{bx + c}{\sqrt{x^2 + bx + c} + x} > \frac{b}{2} \Leftrightarrow bx + c \\ &> \frac{b}{2}\sqrt{x^2 + bx + c} + \frac{b}{2}x \Leftrightarrow \frac{b}{2}x + c > \frac{b}{2}\sqrt{x^2 + bx + c} \Leftrightarrow bcx + c^2 \\ &> \frac{b^2}{4}(bx + c) \Leftrightarrow c(bx + c) > \frac{b^2}{4}(bx + c) \Leftrightarrow (bx + c)\left(c - \frac{b^2}{4}\right) > 0. \end{aligned} \tag{10}$$

Next, we consider $f(x)$ when $x \to \infty$. Since $f(x) = \frac{b + (c/x)}{\sqrt{1 + (b/x) + (c/x^2)} + 1}$, $f(x) \to \frac{b}{2}$ when $x \to \infty$. Thus, the inferior value happens when $x \to \infty$.

For case (c), if $a = 1$ and $b = 2\sqrt{c}$, $f(x)$ is a constant function with $f(x) \equiv \sqrt{c}$. Thus, every positive point is considered as the optimal solution, which violates our goal of finding a unique minimum solution. Consequently, we derive the second condition

$$a > 1. \tag{11}$$

We rewrite Equation (4) as

$$\sqrt{ax^2 + bx + c} = x + f(x) \tag{12}$$

and take square on both sides. We arrange the expression in the descending order of x and treat $f(x)$ as a constant term for the moment. Through the square for x, it implies