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Logistic map with memory from economic model

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ABSTRACT

A generalization of the economic model of logistic growth, which takes into account the effects of memory and crises, is suggested. Memory effect means that the economic factors and parameters at any given time depend not only on their values at that time, but also on their values at previous times. For the mathematical description of the memory effects, we use the theory of derivatives of non-integer order. Crises are considered as sharp splashes (bursts) of the price, which are mathematically described by the delta-functions. Using the equivalence of fractional differential equations and the Volterra integral equations, we obtain discrete maps with memory that are exact discrete analogs of fractional differential equations of economic processes. We derive logistic map with memory, its generalizations, and "economic" discrete maps with memory from the fractional differential equations, which describe the economic natural growth with competition, power-law memory and crises.

1. Introduction

The logistic differential equation was initially proposed in the population growth model by Verhulst [1]. In this model the rate of reproduction is directly proportional to the product of the existing population and the amount of available resources. This differential equation is actively used in economic growth models (for example, see [2,3]). The logistic map is considered as a discrete analog of this differential equation. The logistic map, which is a simple quadratic map, demonstrates complicated dynamics, which can be characterized as universal and chaotic [4–6].

The logistic differential equation can be derived from economic model of natural growth in a competitive environment. The economic natural growth models are described by equations in which the margin output (rate of output growth) is directly proportional to income. In the description of economic growth the competition effects are taken into account by considering the price as a function of the value of output. Model of natural growth in a competitive environment is often called a model of logistic growth. We first describe the model of logistic growth, which does not take into account the memory effects.

Let $Y(t)$ be a function that describes the value of output at time t. We assume that all manufactured products are sold (the assumption of market unsaturation). Let $I(t)$ be a function that describes the investments made in the expansion of production, that is, the value of $I(t)$ is the difference between the total investment and depreciation costs. In the model of natural growth, it is assumed that the marginal value of output $(dY(t)/dt)$ is directly proportional to the value of the net investment $I(t)$. As a result, we can use the accelerator equation

$$\frac{dY(t)}{dt} = \frac{1}{v} \cdot I(t), \quad (1)$$

where v is a positive constant that is called the accelerator coefficient, $1/v$ is the marginal productivity of capital (rate of accelera-

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tion), and $dY(t)/dt$ is the first order derivative of the function $Y(t)$ with respect to time t.

In the logistic growth model the price $P(t)$ is considered as a function of released product $Y(t)$, i.e. $P=P(Y(t))$. The function $P=P(Y)$ is usually considered as a decreasing function, that is, the increase of output leads to a decrease of price due to market saturation.

Assuming that the amount of net investment is a fixed part of the income $P\cdot Y(t)$, we get

$$I(t) = m \cdot P \cdot Y(t), \tag{2}$$

where m is the norm of net investment $(0 < m < 1)$, specifying the share of income, which is spent on the net investment.

Substituting (2) into Eq. (1), we obtain

$$\frac{dY(t)}{dt} = \frac{m}{v} \cdot P(Y(t)) \cdot Y(t). \tag{3}$$

Differential Eq. (3) describes the economic model of natural growth in a competitive environment.

It is often assumed that the price as a function of output $Y(t)$ is linear, i.e. $P(Y(t)) = b - a\cdot Y(t)$, where b is the price, which is independent of the output and a is the margin price. In this case, Eq. (3) has the form

$$\frac{dY(t)}{dt} = \frac{m}{v} \cdot (b - a \cdot Y(t)) \cdot Y(t). \tag{4}$$

Eq. (4) is the logistic differential equation, i.e. the ordinary differential equation of first order that describes the logistic growth. For $a=0$, Eq. (4) describes the natural growth in the absence of competition.

The logistic growth model, which is described by Eq. (4), and the model of the natural growth in a competitive environment, which is described by Eq. (3), implies that the net investment and the marginal output are connected by the accelerator Eq. (1). Eqs. (1), (3), and (4) contain only the first-order derivative with respect to time. It is known that the derivative of the first order is determined by the properties of differentiable functions of time only in infinitely small neighborhood of the point of time. As a result, the models, which are described by Eqs. (3) and (4), assume an instantaneous change of marginal output when the net investment changes. This means not only neglecting the delay (lag) effects, but also the neglect of the memory effects, i.e. the neglect of dependence of output at the present time on the investment changes in the past. In other words, the model of logistic growth (4) does not take into account the effects of memory and delay.

2. Memory effect in economic process

The concept of memory is actively used in econometrics [7,8]. We consider the concept of memory to describe economic processes by analogy with the use of this concept in physics [9, p. 394–395]. The term “memory” means the property that characterizes a dependence of the process state at a given time $t=T$ from the process state in the past $(t<T)$. Economic process with memory is a process, for which the economic indicators and factors (endogenous and exogenous variables) at a given time depend not only on their values at that time, but also on their values at previous time instants from a finite time interval.

A memory effect is manifested in the fact that for the same change of the economic factor, the corresponding dependent economic indicator can vary in different ways, that leads us to the multivalued dependencies of indicators on factors. The multivalued dependencies are caused by the fact that the economic agents remember previous changes of this factor and indicator, and therefore can already react differently. As a result, identical changes in the present value of the factor may lead to the different dynamics of economic indicators.

To describe power-law memory we can use the theory of derivatives and integrals of non-integer order [10–13]. There is an economic interpretation of the fractional derivatives [14,15]. To take into account the effects of power-law memory, the concept of marginal values of non-integer order [16,17] and the concept of the accelerator with memory have been proposed [18,19]. In mathematics different types of fractional-order derivatives are known [10–12]. We will use the left-sided Caputo derivative with respect to time. One of the main distinguishing features of the Caputo fractional derivatives is that the action of these derivatives on a constant function gives zero. Using only the left-sided fractiona-order derivative, we take into account the history of changes of economic indicators and factors in the past. The economic process at time $t=T$ can depend on changes in the state of this process in the past, that is for $t<T$. The right-sided Caputo derivatives are defined by integration over $t>T$. In order to have correct dimensions of economic quantities we will use the dimensionless time variable t.

The left-sided Caputo derivative of order $\alpha>0$ is defined by the formula

$$\left(D^{\alpha}_{0+}Y\right)(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{Y^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}, \tag{5}$$

where $\Gamma(\alpha)$ is the gamma function, $Y^{(n)}(\tau)$ is the derivative of the integer order $n:=[\alpha]+1$ of the function $Y(\tau)$ with respect to the variable τ: $0<\tau<t$. For the existence of the expression (5), the function $Y(\tau)$ must have the integer-order derivatives up to the $(n-1)$th-order, which are absolutely continuous functions on the interval [0, t]. For integer orders $\alpha=n$ the Caputo derivatives coincide with the standard derivatives [11, p. 79], [12, p. 92–93], i.e. $(D^n_{0+}Y)(t) = Y^{(n)}(t)$ and $(D^0_{0+}Y)(t) = Y(t)$.

The generalization of the standard accelerator Eq. (1), which takes into account the memory effects of the order α, can be given [18] in the form

$$\left(D^{\alpha}_{0+}Y\right)(t) = \frac{1}{v} \cdot I(t), \tag{6}$$

where $v = 1 / M$. For $\alpha=1$ Eq. (6) takes the form (1).

Note that the accelerator Eq. (6) includes the standard equation of the accelerator and the multiplier, as special cases [18]. This can be seen by considering Eq. (6) for $\alpha=0$ and $\alpha=1$. Using the property $(D^1_{0+}X)(t) = X^{(1)}(t)$ of the Caputo derivative, formula (6) with $\alpha=1$ takes the form of Eq. (1) that describes the standard accelerator. Using $(D^0_{0+}Y)(t) = Y(t)$, Eq. (6) with $\alpha=0$ is written as $I(t)=v\cdot Y(t)$, which is the equation of standard multiplier. Therefore, the accelerator with memory, given by Eq. (6), generalizes the concepts of the standard multiplier and accelerator [18].

3. Equation of logistic growth with memory and crises

To take into account the power-law memory effects in the natural growth model with a competitive environment, we use Eq. (6), which describes the relationship between the net investment and the margin output of non-integer order [16,17]. Substituting expression (2), where $P=P(Y(t))$, into Eq. (6), we obtain

$$\left(D^{\alpha}_{0+}Y\right)(t) = \frac{m}{v} \cdot P(Y(t)) \cdot Y(t), \tag{7}$$

where $(D^{\alpha}_{0+}Y)(t)$ is the Caputo derivative (5) of the order $\alpha \geq 0$ of the function $Y(t)$ with respect to time. Eq. (7) is the so-called fractional differential equation with derivative of the order $\alpha>0$, [11–13]. The model of natural growth in a competitive environment, which is based on Eq. (7), takes into account the effects of memory with power-law fading of the order $\alpha \geq 0$. For $\alpha=1$, Eq. (7) takes the form of Eq. (3), which describes a model of natural growth in a competitive environment without memory effects.