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Uncertainty quantification for linear hyperbolic equations with stochastic process or random field coefficients

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ABSTRACT

In this paper hyperbolic partial differential equations with random coefficients are discussed. Such random partial differential equations appear for instance in traffic flow problems as well as in many physical processes in random media. Two types of models are presented: The first has a time-dependent coefficient modeled by the Ornstein–Uhlenbeck process. The second has a random field coefficient with a given covariance in space. For the former a formula for the exact solution in terms of moments is derived. In both cases stable numerical schemes are introduced to solve these random partial differential equations. Simulation results including convergence studies conclude the theoretical findings.

1. Introduction

Hyperbolic partial differential equations with random data have been an active research field over the last decades. In ample situations measurements are not accurate enough to allow an exact description of a physical phenomena by a deterministic model. Uncertainty may then be introduced in the appropriate parameters and the distribution of the (now stochastic) solutions is studied. As en example, hyperbolic partial differential equations with random coefficients are applied in the modeling of underground water flow in porous media or, more general, of transport processes in non-uniform media, in the modeling of pollution spread and heat transfer and in traffic simulations. Those types of phenomena can be modeled by hyperbolic conservation laws that have the general from

$$u_t + f(x,t,u)_x = 0 \tag{1.1}$$

in one spatial dimension, i.e., $x \in D \subset \mathbb{R}$. As mentioned, in many *realistic* applications it is often the case that there are uncertainties in the parameters of the function f, or that uncertainty is even intrinsic to the problem. One way to model this is the following. Given a probability space $(\Omega, \mathcal{F}, P)$ we can incorporate those uncertainties by considering the equation

$$\begin{aligned} u_t(x,t,\omega) + f(x,t,\omega,u(x,t,\omega))_x &= 0, \\ u(x,0,\omega) &= g(x), \end{aligned} \tag{1.2}$$

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where f is a (in general nonlinear) function that now depends not only on space, time, and the unknown function u, but also on a stochastic variable $\omega \in \Omega$ that accounts for the uncertainties in the parameters of the conservation law. A random function $u : D \times [0,T] \times \Omega \to \mathbb{R}$ for which Equation (1.2) holds P-almost everywhere in Ω (that is almost surely) is called a (strong) solution. We are then interested in the distribution or in the evolution of certain moments of the solution of this equation, typically of the expectation $\mathbb{E}(u)$ and the variance $\mathbb{V}(u)$.

We restrict our attention to linear advection equations with a random transport velocity as a prototype problem. There are many results in literature for hyperbolic equations with coefficients that are (real-valued) random variables, i.e. which do not depend on space or time. For instance, the authors in [17,5,3] present both theoretical results and numerical approximations. In [6] the authors present expressions for the distribution of the solution of a linear advection equation with a time-dependent velocity, given in terms of the probability density function of the underlying integral of the stochastic process. Concrete results are presented in the case where the velocity field is deterministic, a random variable and Gaussian. Further, the same authors introduce numerical schemes for the mean of the solution of the linear transport equation with homogeneous random velocity and random initial conditions in [4] and the authors in [7] extend the setting to Gaussian processes and telegraph processes. In [13] the linear advection equation with space- and time-dependent coefficients are subject of research. The authors develop numerical methods using polynomial chaos to solve the advection equation with a transport velocity given by a Gaussian or a log-normal distribution. In [1] we applied similar methods, like the ones developed here, to the magnetic induction equation and linear acoustics, both with a time- and space-dependent random background velocity field.

In order to approximate the moments of equations of type (1.2) numerically, methods are either based on a Monte Carlo approach or use a stochastic Galerkin or, more general, a polynomial chaos approach (see [13,11,19] and references therein). The latter approach is not suitable for any distribution. So far this approach is limited to uniform or Gaussian distributed fields or processes. A Monte Carlo method, on the other hand may also be used when dealing with jump processes or Lévy random fields. This comes, however, to the price of a lower convergence rate of the Monte Carlo method. We point out that a more efficient multilevel Monte Carlo approximation could be used in this article, but we refrain from doing so, since we wish to focus on the numerical approximation in the temporal and spatial domain as well as the approximation of the coefficient. For a result on the convergence and computational complexity of the multilevel Monte Carlo approximation for general Hilbert-space-valued random variables we refer to [2]. For a multilevel Monte Carlo finite volume method see for instance [16]. A further advantage of a Monte Carlo method based approximation is that it is non-intrusive, meaning that already implemented numerical solvers can be readily used. In addition, it does not depend on the correlation length of the stochastic input, leading to a large number of Karhunen–Loève terms for weakly correlated fields.

The article is structured as follows. In the first section we examine the linear transport equation with a time dependent coefficient $a = (a(t), t \in [0,T])$ given by the Ornstein–Uhlenbeck process. We derive a closed form expression for the moments of the distribution of the solution. We thereby extend the result found in [17] and [6]. Furthermore, we introduce a second order (in space and time) Monte Carlo method to approximate the solution. We present simulation results and a convergence study. The last section presents the linear transport equation with a space-dependent coefficient $a = (a(x), x \in D)$, assumed to be a Gaussian/Lévy random field over the domain D. Here, we also present a second order (in space and time) Monte Carlo method for the approximation of the solutions. We show simulations and a self-convergence study. Although, in both cases the random transport equation is scalar and linear, we see interesting effects in the moments of the solution that differ from the deterministic variants. Furthermore, the numerical methods/discretizations for the approximation of moments of the solution to the equations become non-trivial.

2. Time-dependent uncertainty modeled by the Ornstein–Uhlenbeck process

In this section we are concerned with the distribution of the solution to the random partial differential equation

$$\begin{aligned} u_t(x,t,\omega) + (a(t,\omega)\, u(x,t,\omega))_x &= 0, \\ u(x,0,\omega) &= g(x) \end{aligned} \tag{2.1}$$

where we model uncertainty in a way that allows for changes over time. That means we want to solve an advection equation with a time-dependent stochastic advection parameter. Let us start by defining $a = (a(t), t \in [0,T])$ as the solution of the Ornstein–Uhlenbeck process

$$\begin{aligned} da(t) &= \theta(\mu - a(t))dt + \sigma\, dW(t), \\ a(0) &= a_0, \end{aligned} \tag{2.2}$$

where $W = (W(t), t \in [0,T])$ is a standard Brownian motion and $\mu \in \mathbb{R}$, $\theta > 0$ and $\sigma > 0$ are parameters. In general the initial condition can be random as well. A standard Brownian motion or Wiener process, defined on the probability space $(\Omega, \mathcal{F}, P)$, is a continuous stochastic process which starts in zero P-a.s and has independent and normally distributed increments, i.e., $W_t - W_s \sim \mathcal{N}(0, t-s)$. The idea of equation (2.2) is that there are two competing features, one is the introduction of noise via the process W, the other is the relaxation of the solution to the mean (see Fig. 1(c) for some sample solutions). For every $t \in [0,T]$ the random variable $a(t)$ is normally distributed with mean and variance