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A time-fractional generalised advection equation from a stochastic process

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ABSTRACT

A generalised advection equation with a time fractional derivative is derived from a continuous time random walk on a one-dimensional lattice, with power law distributed waiting times. We consider walks governed by a two-sided jump density and walks governed by a one-sided jump density. With the two-sided density, the particle can jump in both directions on the lattice, whereas with the one-sided density the particle cannot jump in one of these directions. The master equations describing the evolution of the probability density for the position of the particle are different for each of the jump densities. However in an advective limit both master equations limit to a common generalized advection equation with time fractional derivatives.

We have also considered the stochastic processes in a discrete time setting, again arriving at different discrete time master equations for each of the jump densities. The discrete time master equations can be used to provide different numerical approximations to the solutions of the fractional generalized advection equation. The approximations allow us to compare the efficacy of the one-sided and two-sided densities.

1. Introduction

Continuous time random walks (CTRWs) [1,2], and more recently discrete time random walks (DTRWs) [3–5], have been widely employed as physically consistent stochastic processes whose master equations limit to fractional order partial differential equations (fPDEs). Examples include time fractional Fokker–Planck equations [6–10], fractional reaction diffusion equations [11–14], fractional cable equations [15–17] and fractional advection dispersion relations [7,18]. Fractional order ordinary differential equations have also been derived by considering CTRWs through compartments [19–21]. The temporal fractional order derivatives that arise in CTRW formulations arise from power law distributed waiting times between jumps. Spatial fractional order derivatives have also been motivated in CTRW formulations by considering power law distributed jumps lengths [7,22–27]. Space, and time, fractional derivatives for transport on networks, including finite and infinite lattices, have also been considered [28–32].

The solutions of time fractional differential equations that can be derived from CTRWs or DTRWs share important physical properties of the probability density governing the random walks, such as positivity and boundedness [4,11]. Furthermore, in contrast to fractional order models where fractional derivatives have been introduced in an *ad hoc* way to replace integer order derivatives, there is no ambiguity about the units of the constants that arise in the derivations from an underlying stochastic process [20]. The generalized master equations for the DTRW formulations can also be employed to provide numerical solutions that approximate the solutions of the corresponding limiting partial differential equations or ordinary differential equations [3,4,19].

In this work we have derived a time fractional partial differential advection equation by considering an advective limit in the generalized master equations for CTRWs on a one dimensional lattice with power law distributed waiting times. We have also derived the generalized master equations for corresponding DTRWs which limit to the same fractional advection equation and we use these master equations as a basis for obtaining numerical approximations to the solutions of the fractional advection equation.

The remainder of this paper is as follows: In Section 2 we derive the generalized master equations for CTRWs on a one-dimensional lattice, with a power law waiting time density and

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with two different jump length densities - a two-sided density, and a one-sided density. We derive a fractional Fokker–Planck equation from the diffusive limit of the master equation, and a fractional generalized advection equation from an advective limit of the master equation. In Section 3 we derive the generalized master equations for DTRWs on a one-dimensional lattice with a power law waiting time probability mass function and with two different jump length densities. We show that the fractional generalized advection equation is recovered in an advective limit. In Section 4 we consider two different numerical approximations for the solution of the fractional advection equation, one based on the DTRW master equation with a two-sided jump length density and the other based on the DTRW master equation with a one-sided jump length density. This is illustrated with an example. We conclude with a short discussion in Section 5.

2. The master equation of a CTRW

The CTRW on a one-dimensional lattice is a stochastic process in which a particle resides on a lattice site for some random amount of time, drawn from a waiting time probability density function, before jumping to a site on the lattice governed by a jump length probability density. The stochastic CTRW process has been widely employed in derivations of fractional Fokker–Planck type equations [6,8–10,14]. There are two fundamental steps in these derivations. The first is the derivation of the generalized master equation that governs the time evolution of the probability density for the location of the particle. The second is taking the diffusive limit of the generalized master equation to obtain a partial differential equation.

For completeness, we revisit the derivation of the generalized master equation. We also include the consideration of two different jump length densities; a two-sided density, and a one-sided density. We then consider different limits to fractional partial differential equations; a diffusive limit and an advective limit.

To begin, we consider a one-dimensional lattice with sites denoted by x_i where $i \in \mathbb{N}$. The flux of probability of the particle entering the lattice site x_i at time t, after having taken n jumps can be defined recursively by,

$$q_{n+1}(x_i,t) = \sum_j \int_0^t \Psi(x_i,t|x_j,t')q_n(x_j,t')dt', \tag{1}$$

where $\Psi(x_i, t|x_j, t')$ is the transition probability density for a particle that arrived at lattice site x_j at time t' to jump to lattice site x_i at time t. As Ψ is independent of n, the number of jumps taken, we may write the flux entering lattice site x_i, unconditional on n, as,

$$q(x_i,t) = \sum_{n=0}^{\infty} q_n(x_i,t). \tag{2}$$

The flux entering the lattice site x_i after any number of steps can then by written recursively as,

$$q(x_i,t) = q_0(x_i,t) + \sum_j \int_0^t \Psi(x_i,t|x_j,t')q(x_j,t')dt'. \tag{3}$$

In the following we suppose that Ψ is separable such that,

$$\Psi(x_i,t|x_j,t') = \lambda(x_i,t|x_j)\psi(t-t'). \tag{4}$$

Here ψ is a waiting time density that governs how long the particle will stay at the site, and λ a jump length density that governs the length of the jump. The jump length density is normalised such that,

$$\sum_i \lambda(x_i,t|x_j) = 1, \tag{5}$$

and the waiting time density is normalised as,

$$\int_0^{\infty} \psi(t)dt = 1. \tag{6}$$

In the case where the particle begins at a lattice site, x_0, at time $t = 0$, the initial flux condition will be a product of a Kronecker and a Dirac delta functions, i.e. $q_0(x_i, t) = \delta_{x_i,x_0}\delta(t)$. Other initial conditions have been considered recently in [33]. For the subsequent derivation, we split the flux into the discontinuous and differentiable components, i.e.,

$$q(x_i,t) = \delta_{x_i,x_0}\delta(t) + q^+(x_i,t), \tag{7}$$

where the differentiable component is given by,

$$q^+(x_i,t) = \sum_j \int_0^t \lambda(x_i,t|x_j)\psi(t-t')q(x_j,t')dt'. \tag{8}$$

The master equation governs the evolution of the probability density, $\rho(x_i, t)$, for the position of the particle. This probability density is related to the flux, via

$$\rho(x_i,t) = \int_0^t \Phi(t-t')q(x_i,t')dt', \tag{9}$$

where Φ is the survival function associated with the waiting time density. The survival function can be computed from the waiting time density,

$$\Phi(t) = 1 - \int_0^t \psi(t')dt'. \tag{10}$$

To obtain the master equation we first differentiate Eq. (9) to give,

$$\begin{aligned}\frac{\partial\rho(x_i,t)}{\partial t} &= q^+(x_i,t) - \int_0^t \psi(t-t')q(x_i,t')dt', \\ &= \sum_j \int_0^t \lambda(x_i,t|x_j)\psi(t-t')q(x_j,t')dt' \\ &\quad - \int_0^t \psi(t-t')q(x_i,t')dt'.\end{aligned} \tag{11}$$

It remains to express the right hand side of this equation in terms of ρ. This can be achieved by introducing a memory kernel $K(t)$ with the property that

$$\int_0^t \psi(t-t')q(x_i,t')dt' = \int_0^t K(t-t')\rho(x,t')dt'. \tag{12}$$

An explicit representation of the memory kernel can be obtained using Laplace transform methods. We use the notation

$$\mathcal{L}_t\{g(x,t)\} = \int_0^{\infty} e^{-st}g(x,t)dt. \tag{13}$$

for the Laplace transform from t to s and $\mathcal{L}_s^{-1}$ as the inverse Laplace transform from s to t. We now take the Laplace transforms of Eq. (9) and of Eq. (12) using the convolution theorem, and we combine the results to obtain the Laplace transform of the memory kernel,

$$\mathcal{L}_t\{K(t)\} = \frac{\mathcal{L}_t\{\psi(t)\}}{\mathcal{L}_t\{\Phi(t)\}} \tag{14}$$

and then the memory kernel is given by,

$$K(t) = \mathcal{L}_s^{-1}\left\{\frac{\mathcal{L}_t\{\psi(t)\}}{\mathcal{L}_t\{\Phi(t)\}}\right\}, \tag{15}$$

The master equation for the CTRW is now simply found by substituting Eq. (12) into Eq. (11). This yields

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