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Explicit Boundary Controls for Finite Diffusion Process

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Abstract: Three types of boundary controls are considered: proportional control, feed-forward control and integrated-by-past-values control. They stabilise the diffusion process exponentially at the desired reference on the controlled boundary. The process is actuated through the Neumann boundary while on other boundaries it is given with the Dirichlet data and, in higher dimensions, also with the homogeneous Neumann boundary condition. Two systems, deterministic and stochastic are compared as an ideal and real description of a physical system. Disturbance in the stochastic system is originated by a white noise on the controlled boundary. Physically this noise represents different types of uncertainties like a side-reaction mass-flux and other uncertainties that are not part of a deterministic model. At first the proportional control is analysed on the deterministic system and then it is represented in the feedback integrated past controls form, and lastly the developed controls are modified as a version of stochastic control that applied on the stochastic system, which boundary is disturbed with an unmeasured noise. The regulation error that emerges in the stochastic system control is analysed and proved to be a zero-mean, bounded-variance Gaussian variable that is correlated spatially and temporarily. The process control is demonstrated on an electrodeposition process example.

Keywords: Distributed parameter system, stochastic process, boundary control

1. Introduction

The purpose of the paper is to propose and analyse exact controls motivated by applications in electrochemistry of batteries (Tenno and Nefedov, 2014), electroplating (Tenno and Pohjoranta 2014; Pohjoranta and Tenno, 2014) and other oxidation processes where diffusion mass transfer is important and controlled to prevent the ion depletion of the electrolyte or other simpler mass transfer processes like in control of the desalination plant (Nguyen and Tenno, 2017). This paper proves that these controls are stabilizing controls. The proportional feedback control is represented in two equivalent forms as the feed-forward control and integrated-by-past-values control that is more practical as the latter control do not require measurements on the boundary. This control is the right feedback form for a deterministic system and for a stochastic system if the control is modified for the noisy boundary. In applications, the stochastic system emerges from the model uncertainty. For example, from the surface chemistry of real processes that is understood incompletely (Newman *et al*, 2004). Any model that deals with surface chemistry includes, to some extent, uncertainties. Consequently, a stochastic model should be considered that deals with uncertainties in the form of unmeasured noises. The stochastic model, if applied, should stabilise a process even if a real mass flux that depends on side reactions is modeled as a noise on the boundary that cannot be observed or controlled precisely. The mass flux on average must be controlled on the bases of a stochastic model. This creates a regulation error in the subdomain interior and on the boundary as a difference between the stochastic process and the deterministic process. This paper proves that the regulation error on the boundary has zero mean and bounded variance and is a Gaussian random variable that is strongly correlated temporarily. This error enters the subdomain where it induces a secondary regulation error that is bounded, correlated spatially and temporarily. This proves that in PDEs control slower damping of the regulation error takes place, which is a different property from ODE systems.

2. The problem statement

In this section, a stochastic diffusion process is considered that is controlled through the noisy Neumann boundary with the proportional control and integrated-by-past-values control. The latter control applied afterwards on the stochastic and mean value processes.

The problem statement. Let $t \in [0,\infty)$, $L > 0$ and Ω be a bounded subdomain in R^d that is a *d*-dimensional hypercube $\Omega = (0, L)^d$. The boundaries of the cube are divided into three sets: Γ_D , Γ_N , and Γ_0 such that their union is the natural subdomain boundary $\Gamma_D \bigcup \Gamma_N \bigcup \Gamma_0 = \partial\Omega$ without overlapping of boundaries: $\Gamma_D \bigcap \Gamma_N = 0$, $\Gamma_D \bigcap \Gamma_0 = 0$, and $\Gamma_N \bigcap \Gamma_0 = 0$. In $Q = \Omega \times [0,\infty)$, diffusion process (1) with constant diffusivity $D > 0$ is considered.

$$c_t = D\Delta c \qquad \text{in } Q \tag{1}$$

In (1), c_t is the time derivative and Δc is the Laplacian of *c*. The process $c(t, x)$ is set initially on a constant level $c(0, x) = c_b$ in Ω . Equation (1) satisfies the mixed boundary

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conditions: the Dirichlet condition $c = c_b$ on boundary Γ_D and the Neumann conditions on other boundaries Γ_N and Γ_0. The control $u(t)$ acts through the Neumann boundary Γ_N.

$$-D\nabla c \cdot \mathrm{n} = u(t,\omega) + \sigma \dot{W}(t) \quad \text{on } \Gamma_N \times [0,\infty) \tag{2}$$

In (2), ∇c is the gradient of c. The symmetry (or insulation) condition is assumed on other boundaries of Γ_0 (i.e., in between Γ_D and Γ_N).

$$-D\nabla c \cdot \mathrm{n} = 0 \quad \text{on } \Gamma_0 \times [0,\infty) \tag{3}$$

In (2) and (3), n denotes the normal to the controlled boundary Γ_N and insulating boundary Γ_0 correspondingly. The controlled boundary Γ_N is a noisy boundary modelled with a generalised white noise $\dot{W}(t)$ that gains its exact meaning in a weak-form representation through Ito's integral, where it appears as a finite dimensional Wiener process $W(t)$ in R^m, $m \geq d$. This noise is applied point-wise along the boundary Γ_N. The coordinates $W_i(t)$ of Wiener process $W(t)$ are given in a canonical probability space (Ω, F, P) that is a fixed space once and for ever. $W_i(t)$ is a standard Wiener process with variance $\mathbf{M}W_i^2(t) = t$. In (2), σ is the diagonal $d \times m$ matrix that stands for the variance of noise intensity; ω is the outcome of the sampled space Ω where all the noise dependent stochastic processes are defined.

This noise on the boundary represents ignored physical processes, like unmodelled secondary or surface reactions in electroplating or some other unmeasured in practice process that must be supressed with an effective control (yet to be found).

This system is assumed to be observed on the Dirichlet boundary Γ_D as the constant c_b and on the noisy boundary Γ_N as the stochastic control $u(t,\omega)$ in combination with noise; the noise $\sigma\dot{W}(t)$ cannot be measured. In (2), a stochastic control $u(t,\omega) + \sigma\dot{W}(t)$ is applied in a way how real system (1)-(3) it senses. We designate this control with the shorthand $u_r(t)$ of (4).

$$u_r(t) = u(t,\omega) + \sigma\dot{W}(t) \tag{4}$$

The system (1)-(3) without boundary noise coincides with the system for mean value process $\mathbf{M}c = m$ that satisfies (5)

$$m_t = D\Delta m \qquad \text{in } Q \tag{5}$$

and $m(0,x) = c_b$ in Ω. For $\mathbf{M}u(t,\omega) = u(t)$, this system (5) is deterministic in the subdomain and on all boundaries.

$$m = c_b \qquad \text{on } \Gamma_D \times [0,\infty) \tag{6}$$

$$-D\nabla m \cdot \mathrm{n} = u(t) \quad \text{on } \Gamma_N \times [0,\infty) \tag{7}$$

$$-D\nabla m \cdot \mathrm{n} = 0 \qquad \text{on } \Gamma_0 \times [0,\infty) \tag{8}$$

We analyse two specific controls motivated by applications in electrochemistry. These are the proportional control

$$u(t) = -K_p\left(m(t,0) - c_d\right) \tag{9}$$

and integrated-by-past-values control

$$u(t) = -K_p\left(c_b - c_d\right) - K_P\int_0^t \frac{u(\tau)}{\sqrt{\pi(t-\tau)D}}\left(1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-\frac{n^2L^2}{(t-\tau)D}}\right)d\tau \, . \tag{10}$$

In (9) and (10), K_p is the control gain, $K_p > 0$ and c_d is the control reference.

In stochastic realisation, the former control (10) is modified as a stochastic version of integrated control (11) over the past noisy values of controls $u_r(\tau) = u_r(\tau,\omega)$, $0 \leq \tau \leq t$

$$u(t,\omega) = -K_p\left(c_b - c_d\right) - K_P\int_0^t \frac{u_r(\tau,\omega)}{\sqrt{\pi(t-\tau)D}}\left(1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-\frac{n^2L^2}{(t-\tau)D}}\right)d\tau \, . \tag{11}$$

Section 3 states that these controls (9), (10) and (11) stabilize the deterministic process (5)-(8) on the boundary $m(t,0)$ at the desired reference c_d and that the regulation error $c(t,0) - m(t,0)$ of the stochastic process (1)-(4) is unbiased, bounded variance, and Gaussian. The regulation error is correlated temporarily.

The reduced control problem. In dependence on the desired results, two approaches can be applied. The first one is to extend the problem with complex geometry of the subdomain where the process lives and with the functional space noises on the boundary and in the subdomain. In this statement the general theory (Da Prato and Zabczyk, 1992; 1993) provides tools for qualitative analysis. The second approach is to diminish the problem to the case where exact solution of the problem (control law) can be obtained. In control practice, one is more interested in exact results than in qualitative results provided by the general theory for systems with complex geometry and with the functional space noises. In order to have exact analytical results (control law) the problem will be restricted next to the scalar case $d = 1$ and then some more comments will be given later for extension of the results to other simple cases $d > 1$ (e.g., 2D rectangle or 3D cube) that do not require new ideas of extension.

3. The main results

Theorem 1 and Corollaries 1.1-4 apply to the deterministic system (5)-(8) actuated on the boundary (7) with control (9).