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Robust Stability Analysis of Discrete-Time Linear Systems with Dynamics Determined by a Markov Process ⋆

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Abstract: In this paper, we deal with discrete-time linear stochastic systems whose state transition is described by a sequence of random matrices determined by a Markov process. For such systems, we first show a linear-matrix-inequality-based stability condition. The use of conditional expectation plays a key role in the derivation. Then, we assume that the systems are uncertain in the sense that they are described by a sequence of random polytopes determined by a Markov process (to which the actual underlying random matrix sequence should belong), and extend the result about the above uncertainty-free case toward robust stability analysis, which leads to our main result. To show the usefulness of the result, we apply it to the case with stochastic switched systems having uncertain coefficient matrices, and provide a numerical example.

Keywords: Stochastic systems, Markov processes, LMI, random polytopes

1. INTRODUCTION

When we consider applying control theory to real systems, their models are generally required. In particular, if the systems include inevitable stochastic factors but if it is insufficient in the modeling to merely consider stochastic disturbance inputs in the system representation, we might realize that the dynamics of the systems themselves should be regarded as stochastic. Such systems having stochastic dynamics are called random dynamical systems (Arnold, 1998). In the field of random dynamical systems, their behaviors and properties have been studied, e.g., in Yu et al. (1990); Arnold and Chueshov (1998); Wang (2012). However, the number of studies dealing with these systems as the targets for control is still limited. The ultimate purpose of our research is to develop an analysis and synthesis framework for a class of such systems from the viewpoint of control engineering, through restricting our attention to the discrete-time linear case. In particular, this paper deals with discrete-time linear stochastic systems whose state transition is described by a sequence of random matrices determined by a Markov process (Knill, 2009).

In an earlier study (Hosoe and Hagiwara, 2014), the present authors studied a control problem for discrete-time linear stochastic systems whose state transition is described by a sequence of random matrices that are *independently and identically distributed (i.i.d.) with respect to time*. In contrast, the present study corresponds to the case when these matrices do not satisfy this assumption but are actually determined by a (temporally homogeneous) Markov process.

In this paper, we first tackle a stability problem for stochastic systems described by a random matrix sequence determined by a Markov process. We derive a linear-matrix-inequality (LMI)-based sufficient condition for stability of the systems. In particular, the use of conditional expectation, which was needless to employ in the above earlier study, plays a key role in the derivation. Then, we further consider the situation where the way of determining the underlying random matrix sequence with a Markov process is somewhat uncertain (e.g., due to inevitable modeling errors) and describing the systems by a sequence of random polytopes (Hug, 2013) determined by a Markov process (to which the actual underlying random matrix sequence should belong) is reasonable. For such uncertain stochastic systems described by a random polytope sequence determined by a Markov process, we extend the above result about the uncertainty-free case and show a sufficient inequality condition for robust stability as the main result in this paper; we will see that the condition consists only of the inequalities related to the vertices of the random polytopes, as is the case with polytopic uncertainties of the usual deterministic systems (Boyd et al., 1994; Oliveira et al., 2002; Ebihara et al., 2015).

As a special case of stochastic systems described by a random matrix sequence determined by a Markov process, we can consider stochastic switched systems whose coefficient matrices in the state space representation are randomly switched at each time step according to the mode transition on a finite-state (i.e., finite-mode) Markov chain; such systems are also called Markov jump systems (Costa et al., 2005; Souza, 2006; Zhang and Lam, 2010). By applying the above result about robust stability analysis to this special case, we can immediately obtain a robust stability condition for stochastic switched systems having

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uncertain coefficient matrices. Robust stability analysis for this special case was studied in Hosoe and Hagiwara (2016) by applying the result in Hosoe and Hagiwara (2014) after deriving an equivalent i.i.d. counterpart to the systems through extending the dimension of the state space. An advantage of the new direction in this paper over the use of the equivalent i.i.d. counterpart is that we can circumvent the increased conservativeness and computation cost associated with the state space extension (the associated situation will be clearer in the numerical example given later).

The contents of this paper are as follows. In Section 2, we introduce stochastic systems described by a random matrix sequence determined by a Markov process dealt with in this paper (without assuming uncertainties for the moment). For these systems, we derive in Section 3 an LMI-based sufficient stability condition through the use of conditional expectation. Then, Section 4 gives our main result by extending this result to that for robust stability analysis. More specifically, we assume that the random matrices describing the state transition of the systems belong to random polytopes determined by the underlying Markov process, and derive a robust stability condition with respect to the deterministic uncertain time-invariant parameter that determines the relative position of the random matrices in the random polytopes. In Section 5, to show the usefulness of the main result, we further consider the case with stochastic switched systems having uncertain coefficient matrices, and provide a numerical example.

We use the following notation in this paper. $\mathbf{R}$ and $\mathbf{N}_0$ denote the set of real numbers and that of non-negative integers, respectively. $\mathbf{R}^n$ and $\mathbf{R}^{n\times n}$ denote the set of n-dimensional real column vectors and that of $n \times n$ real matrices, respectively. $\mathbf{S}^{n\times n}$ and $\mathbf{S}_+^{n\times n}$ denote the set of $n\times n$ symmetric matrices and that of $n\times n$ positive definite matrices, respectively. $||(\cdot)||$ denotes the Euclidean norm of the vector $(\cdot)$. $E[(\cdot)]$ denotes the expectation of the random variable $(\cdot)$; this notation is also used for the expectation of the random matrix $(\cdot)$.

2. DISCRETE-TIME LINEAR SYSTEMS WITH DYNAMICS DETERMINED BY A MARKOV PROCESS

Let us consider the Markov process ξ, whose value at time $k \in \mathbf{N}_0$ is denoted by $\xi_k \in \mathbf{R}^Z$ and belongs to the given set $\boldsymbol{\Xi}$. By definition, the value of ξ_k determines the (conditional) distribution of the random vector ξ_{k+1} for each $k \in \mathbf{N}_0$ (for more details of Markov processes, see, e.g., Knill (2009)). This paper assumes that ξ is temporally homogeneous, which means that the above distribution is common for all $k \in \mathbf{N}_0$. We denote the associated cumulative distribution function by $\mathcal{F}(\xi_\star^+|\,\xi_\star)$ under the following notation: when ξ_k takes the value $\xi_\star$, the corresponding random vector ξ_{k+1} is denoted by $\xi_\star^+$.

This paper deals with the discrete-time linear system

$$x_{k+1} = A_k x_k, \quad A_k = A(\xi_k) \tag{1}$$

whose state transition is described by the random matrix sequence $(A(\xi_k))_{k\in\mathbf{N}_0}$ determined by the above Markov process ξ, where $x_k \in \mathbf{R}^n$ and $A(\cdot)$ is a matrix-valued function. In the following section, we tackle the problem of deciding whether the above system is globally exponentially stable in the second moment (Kozin, 1969):

Definition 1. The system (1) is said to be globally exponentially stable in the second moment if there exist $a > 0$ and $0 < \lambda < 1$ such that

$$\sqrt{E[||x_k||^2]} \le a||x_0||\lambda^k \quad (\forall k \in \mathbf{N}_0, \forall x_0 \in \mathbf{R}^n). \tag{2}$$

As is obvious from the above definition, the constant λ gives an upper bound of the convergence rate with respect to the sequence $\left(\sqrt{E[||x_k||^2]}\right)_{k\in\mathbf{N}_0}$.

3. STABILITY CONDITION

In this section, we derive a stability condition of the system (1) when $A(\cdot)$ is given.

For $X \in \mathbf{S}_+^{n\times n}$ and $G \in \mathbf{R}^{n\times n}$ such that

$$G + G^T - X > 0, \tag{3}$$

let us first consider a map $T : \boldsymbol{\Xi} \to \mathbf{S}^{n\times n}$ satisfying

$$\begin{bmatrix} \lambda^2 X + T(\xi_\star) & A(\xi_\star)^T G^T \\ * & G + G^T - X \end{bmatrix} \ge 0 \quad (\forall \xi_\star \in \boldsymbol{\Xi}), \tag{4}$$

where $*$ denotes the transpose of the upper right block in the matrix. Such a T is ensured to exist by (3), and is a deterministic map by definition.

Once we apply this map to the random vector $\xi_\star^+$, however, then the resulting $T(\xi_\star^+)$ is a random matrix whose (conditional) distribution is determined by the underlying deterministic vector $\xi_\star$. The following theorem establishes a link between stability of the system (1) and the conditional expectations of this random matrix and constitutes a fundamental technical contribution of this paper.

Theorem 1. If there exist $X \in \mathbf{S}_+^{n\times n}$, $G \in \mathbf{R}^{n\times n}$, $T : \boldsymbol{\Xi} \to \mathbf{S}^{n\times n}$ and $0 < \lambda < 1$ satisfying (3), (4) and

$$E[T(\xi_\star^+)|\,\xi_\star] \le 0 \quad (\forall \xi_\star \in \boldsymbol{\Xi}) \tag{5}$$

then the stochastic system (1) is globally exponentially stable in the second moment.

$E[T(\xi_\star^+)|\,\xi_\star]$ represents the conditional expectation of the random matrix $T(\xi_\star^+)$ under the given event $\xi_\star$ and is a deterministic matrix-valued function in the deterministic $\xi_\star$. Hence, the inequality (5) just requires the negative semidefiniteness of the resulting ($\xi_\star$-dependent) deterministic matrix. The proof of Theorem 1 is given as follows.

Proof. By post- and pre-multiplying $[I, -A(\xi_\star)^T]^T$ and its transpose on (4), respectively, we have

$$\lambda^2 X + T(\xi_\star) - A(\xi_\star)^T X A(\xi_\star) \ge 0. \tag{6}$$

This inequality holds regardless of $\xi_\star \in \boldsymbol{\Xi}$. Hence, each time we take samples of x_k and ξ_k at $k \in \mathbf{N}_0$, we have

$$\lambda^2 x_k^T X x_k + x_k^T T(\xi_k) x_k \ge x_{k+1}^T X x_{k+1} \tag{7}$$

by (1). Since the above inequality holds regardless of the samples, we further have

$$\lambda^2 E[x_k^T X x_k] + E[x_k^T T(\xi_k) x_k] \ge E[x_{k+1}^T X x_{k+1}] \tag{8}$$

for each k. Here, let us denote by $E[(\cdot)|\,\xi_0, \ldots, \xi_{k-1}]$ the conditional expectation of the random variable $(\cdot)$ under the condition that the values of $\xi_0, \ldots, \xi_{k-1}$ are determined. Then, noting that [1] $E[(\cdot)] = E[E[(\cdot)|\,\xi_0, \ldots, \xi_{k-1}]]$, we have

[1] For the random variable $(\cdot)$, $E[(\cdot)|\,\xi_0, \ldots, \xi_{k-1}]$ is a function of the random vectors $\xi_0, \ldots, \xi_{k-1}$.