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Integrably bounded set-valued stochastic integrals

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ABSTRACT

The paper is devoted to properties of Aumann and Itô set-valued stochastic integrals, defined as some set-valued random variables. In particular the problem of integrable boundedness of the generalized Itô set-valued stochastic integrals is considered. Unfortunately, Itô set-valued stochastic integrals, defined by E.J. Jung and J.H. Kim in the paper [5], are not in general integrably bounded (see [8,15]). Therefore, in the present paper we consider generalized Itô set-valued stochastic integrals (see [10,11]) defined for absolutely summable and countable subsets of the space $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ of all square integrable $\mathbb{F}$-nonanticipative matrix-valued stochastic processes. Such integrals are integrably bounded and possess properties needed in the theory of set-valued stochastic equations.

1. Introduction

The paper deals with properties of Aumann and Itô set-valued stochastic integrals, defined as some set-valued random variables. Initial studies on Itô set-valued stochastic integrals, defined as subsets of the spaces $\mathbb{L}^2(\Omega, \mathbb{R}^n)$ and $\mathbb{L}^2(\Omega, \mathcal{X})$, have been considered by F. Hiai and M. Kisielewicz (see [2,6,7]), where $\mathcal{X}$ is a Hilbert space. Unfortunately, such defined integrals do not admit their representations by set-valued random variables with values in $\mathbb{R}^n$ and $\mathcal{X}$, because they are not decomposable subset of $\mathbb{L}^2(\Omega, \mathbb{R}^n)$ and $\mathbb{L}^2(\Omega, \mathcal{X})$, respectively. J. Jung and J.H. Kim in [5] have defined the Itô set-valued stochastic integral as a set-valued random variable determined by a closed and decomposable hull of the set-valued stochastic functional integral defined in [6]. Unfortunately, such integrals are not in the general case (see [8,15]) integrably bounded. Therefore, in what follows we shall consider generalized Itô set-valued stochastic integrals (see [10,11]) of absolutely summable countable subsets of the space $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ of square integrable $\mathbb{F}$-nonanticipative matrix-valued stochastic processes defined on a complete filtered probability space $\mathcal{P}_{\mathbb{F}} = (\Omega, \mathcal{F}, \mathbb{F}, P)$. Generalized set-valued stochastic integrals were defined in the paper [11] and some of their properties have been considered in [10]. Let us recall (see [10] and [11]) that

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for a given m-dimensional $\mathbb{F}$-Brownian motion $B = (B_t)_{\geq 0}$ defined on a filtered probability space $\mathcal{P}_{\mathbb{F}}$ and a nonempty subset $\mathcal{G}$ of the space $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$, a generalized Itô set-valued stochastic integral $\int_0^t \mathcal{G}dB_\tau$ is understood as an $\mathcal{F}_t$-measurable set-valued random variable with values in the d-dimensional Euclidean space $\mathbb{R}^d$ and subtrajectory integrals $S_{\mathcal{F}_t}(\int_0^t \mathcal{G}dB_\tau)$ equal to $\overline{\text{dec}}\,\mathcal{J}_t(\mathcal{G})$. By $\mathcal{J}_t$ we denote the Itô isometry with values in the space $\mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{R}^d)$ defined for fixed $t \geq 0$ and every $g \in \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ by setting $\mathcal{J}_t(g) = \int_0^t g_\tau dB_\tau$. Subtrajectory integrals $S_{\mathcal{F}_t}(\int_0^t \mathcal{G}dB_\tau)$ of $\int_0^t \mathcal{G}dB_\tau$ are defined as a set of all $\mathcal{F}_t$-measurable and square integrable selectors of $\int_0^t \mathcal{G}dB_\tau$. It will be also denoted by $S_t(\int_0^t \mathcal{G}dB_\tau)$. In particular, if $\mathcal{G}$ is a nonempty decomposable subset of $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ then $\int_0^t \mathcal{G}dB_\tau = \int_0^t G_\tau dB_\tau$, where $G = (G_t)_{t\geq 0}$ is an $\mathbb{F}$-nonanticipative set-valued process such that $S_{\mathbb{F}}(G) = \text{cl}_{\mathbb{L}}(\mathcal{G})$, where $S_{\mathbb{F}}(G) = \{g \in \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m}) : g_t(\omega) \in G_t(\omega) \text{ for a.e. } (t,\omega) \in \mathbb{R}^+ \times \Omega\}$. In a similar way the Aumann set-valued stochastic integrals can be defined (see e.g. [9], p. 114 and [16]). Namely, for a given $\mathbb{F}$-nonanticipative set-valued stochastic process $F : \mathbb{R}^+ \times \Omega \to \text{Cl}(\mathbb{R}^d)$ such that $S_{\mathbb{F}}(F) \neq \emptyset$, the Aumann set-valued stochastic integral $\int_0^t F_\tau d\tau$ is defined as the set-valued random variable such that $S_t(\int_0^t F_\tau d\tau) = \overline{\text{dec}}\, J_t(S_{\mathbb{F}}(F))$, where J_t denotes (for fixed $t \geq 0$) the mapping with values in the space $\mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{R}^d)$ defined for every $f \in \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^d)$ by setting $J_t(f) = \int_0^t f_\tau d\tau$.

Apart from the above defined Aumann set-valued stochastic integral $\int_0^t F_\tau d\tau$ one can define (see e.g. [9, 12] and [13]) an $(\mathcal{A})$-set-valued stochastic integral $(\mathcal{A})\int_0^t F_\tau d\tau$ by setting $((\mathcal{A})\int_0^t F_\tau d\tau)(\omega) = \int_0^t F(\tau,\omega)d\tau$ for fixed $t \geq 0$ and a.e. $\omega \in \Omega$, where $\int_0^t F(\tau,\omega)d\tau$ denotes the parametrized Aumann integral, i.e., for a.e. fixed $\omega \in \Omega$ the Aumann integral of $F(\cdot,\omega)$. It can be verified (see [9], Lemma 3.1 of Chap. 3 and also [12, 13]) that $(\mathcal{A})\int_0^t F_\tau d\tau$ is an $\mathcal{F}_t$-measurable convex, compact valued set-valued random variable. But for a.e. $\omega \in \Omega$ one has $((\mathcal{A})\int_0^t F_\tau d\tau)(\omega) = \{u(\omega) : u \in J_t(S_{\mathbb{F}}(F))\} \subset \{u(\omega) : u \in \overline{\text{dec}}\, J_t(S_{\mathbb{F}}(F))\} \subset (\int_0^t F_\tau d\tau)(\omega)$. It can be verified (see [9], Corollary 3.1 of Chap. 3) that for every measurable, convex-valued and integrably bounded set-valued process $F : \mathbb{R}^+ \times \Omega \to \text{Cl}(\mathbb{R}^d)$ one has $((\mathcal{A})\int_0^t F_\tau d\tau)(\omega) = (\int_0^t F_\tau d\tau)(\omega)$ for a.e. $\omega \in \Omega$ and $t \geq 0$. For other properties of $(\mathcal{A})\int_0^t F_\tau d\tau$ we refer the reader to [9,12] and [13].

Let us assume that $\mathcal{G} = \{g^n : n \geq 1\} \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$ is absolutely summable, i.e. $\sum_{n=1}^\infty \|g^n\|^2 < \infty$, where $\|\cdot\|$ is a norm of $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$. Then a generalized set-valued integral $\int_0^t \mathcal{G}dB_\tau$ can be defined for every $t \geq 0$ as a set-valued random variable $\mathcal{H}_t : \Omega \to \text{Cl}(\mathbb{R}^d)$ of the form $\mathcal{H}_t(\omega) = \text{cl}\{(\int_0^t g_\tau^n dB_\tau)(\omega) : n \geq 1\}$ for every $\omega \in \Omega$. Indeed, it is clear, that for every $t \geq 0$ a sequence $(\int_0^t g_\tau^n dB_\tau)_{n=1}^\infty$ is the Castaing representation of a set-valued random variable $\mathcal{H}_t$. Furthermore we have $\sup_{n\geq 1} E[|\int_0^t g_\tau^n dB_\tau|^2] \leq \sum_{n=1}^\infty \|g^n\|^2 < \infty$. Then $(\int_0^t g_\tau^n dB_\tau)_{n=1}^\infty$ is a bounded sequence of $\mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{R}^d)$ contained in $S_t(\mathcal{H}_t)$. Therefore, by ([9], Remark 3.6 of Chap. 2), we have $S_t(\mathcal{H}_t) = \overline{\text{dec}}\{\int_0^t g_\tau^n dB_\tau : n \geq 1\} = \overline{\text{dec}}\{\mathcal{J}_t(g^n) : n \geq 1\} = \overline{\text{dec}}\,\mathcal{J}_t(\mathcal{G}) = S_t(\int_0^t \mathcal{G}dB_\tau)$. Then $\mathcal{H}_t = \int_0^t \mathcal{G}dB_\tau$ a.s. for every $t \geq 0$.

Let us also note that for such a set $\mathcal{G} = \{g^n : n \geq 1\}$, a stochastic process $(\sum_{n=1}^\infty |\int_0^t g_\tau^n dB_\tau|^2)_{t\geq 0}$ is a positive submartingale. Firstly, let us observe that such a set $\mathcal{G}$ is bounded in $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$, because $\sup_{n\geq 1} \|g^n\|^2 \leq \sum_{n=1}^\infty \|g^n\|^2 < \infty$. Now, for a fixed $t \geq 0$, let us put $\xi_n^t = \sum_{k=1}^n |\int_0^t g_\tau^k dB_\tau|^2$ where $n \geq 1$ and $\xi^t = \sum_{n=1}^\infty |\int_0^t g_\tau^n dB_\tau|^2$. Let $m = \sum_{n=1}^\infty \|g^n\|^2$. We have $\xi_n^t \leq \xi^t$ a.s. and $E\xi_n^t \leq m < \infty$ for every $n \geq 1$ and $t \geq 0$. Therefore, $\sup_{n\geq 1} E[\mathbb{1}_A \xi_n^t] \to 0$ as $P(A) \to 0$. Then the sequence $(\xi_n^t)_{n=1}^\infty$ of positive random variables converges to ξ^t a.s. for every fixed $t \geq 0$ and it is such that $\lim_{n\to\infty} E[\xi_n^t|\mathcal{F}_s] = E[\xi^t|\mathcal{F}_s]$ a.s. for every $0 \leq s < t < \infty$. On the other hand $\lim_{n\to\infty} E[\xi_n^t|\mathcal{F}_s] = \lim_{n\to\infty} \sum_{k=1}^n E[|\int_0^t g_\tau^k dB_\tau|^2|\mathcal{F}_s] = \sum_{n=1}^\infty E[|\int_0^t g_\tau^n dB_\tau|^2|\mathcal{F}_s]$ a.s. for every $0 \leq s < t < \infty$. Thus $E[\sum_{n=1}^\infty |\int_0^t g_\tau^n dB_\tau|^2|\mathcal{F}_s] = \sum_{n=1}^\infty E[|\int_0^t g_\tau^n dB_\tau|^2|\mathcal{F}_s]$ a.s. for every $0 \leq s < t < \infty$. Finally, by Jensen's inequality we get $E[|\int_0^t g_\tau^n dB_\tau|^2|\mathcal{F}_s] \geq |\int_0^s g_\tau^n dB_\tau|^2$ a.s. for every $n \geq 1$ and $0 \leq s < t < \infty$. Therefore, we have $\sum_{n=1}^\infty |\int_0^s g_\tau^n dB_\tau|^2 \leq E[\sum_{n=1}^\infty |\int_0^t g_\tau^n dB_\tau|^2|\mathcal{F}_s]$ a.s. for every $0 \leq s < t < \infty$.

Let (X, ρ) be a metric space and denote by $\text{Cl}(X)$ a space of all nonempty closed subsets of X. For every $A, C \in \text{Cl}(X)$ let $\bar{h}(A,C) = \sup\{d(a,C) : a \in A\}$, where $d(a,C) = \inf\{\rho(a,c) : c \in C\}$. The Hausdorff distance $h(A,C)$ between $A, C \in \text{Cl}(X)$ is defined by $h(A,C) = \max\{\bar{h}(A,C), \bar{h}(C,A)\}$. It can be verified (see [2], p. 24) that for every sequence $(A_n)_{n\geq 1} \subset \text{Cl}(X)$ converging in the Hausdorff metric

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