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Asymptotics for the determinant of the combinatorial Laplacian on hypercubic lattices

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ABSTRACT

In this paper, we compute asymptotics for the determinant of the combinatorial Laplacian on a sequence of d-dimensional orthotope square lattices as the number of vertices in each dimension grows at the same rate. It is related to the number of spanning trees by the well-known matrix tree theorem. Asymptotics for 2 and 3 component rooted spanning forests in these graphs are also derived. Moreover, we express the number of spanning trees in a 2-dimensional square lattice in terms of the one in a 2-dimensional discrete torus and also in the quartered Aztec diamond. As a consequence, we find an asymptotic expansion of the number of spanning trees in a subgraph of $\mathbb{Z}^2$ with a triangular boundary.

1. Introduction

In this paper we study the asymptotic behaviour of the number of spanning trees in a discrete d-dimensional orthotope square lattice and in the quartered Aztec diamond. Let $L(n_1, \ldots, n_d)$ denote the d-dimensional orthotope square lattice defined by the cartesian product of the d path graphs $P_{n_i}, i = 1, \ldots, d$, where $n_i, i = 1, \ldots, d$, are positive non-zero integers. We set $n_i = \alpha_i n, i = 1, \ldots, d$ and write indifferently n_i or $\alpha_i n$ throughout the paper, where $\alpha_i, i = 1, \ldots, d$, are positive integers. By rescaling the distance between two vertices on the lattice $L(n_1, \ldots, n_d)$ with a factor of $1/n$, the limiting object as n goes to infinity is a d-dimensional orthotope of size $\alpha_1 \times \cdots \times \alpha_d$, that we denote by K_d:

$$K_d := [0, \alpha_1[\times \cdots \times [0, \alpha_d[\,.$$

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The volume of K_d is

$$V_d^d := \prod_{i=1}^{d} \alpha_i.$$

Let $m \in \{1, \dots, d-1\}$, $\{i_q\}_{q=1}^m \subset \{1, \dots, d\}$ and $\{\delta_j\}_{j \in \{1,\dots,d\} \setminus \{i_q\}_{q=1}^m}$ such that $\delta_j \in \{0, \alpha_j\}$. An m-dimensional face of $\overline{K}_d$ is defined by

$$\begin{aligned} &F\left(\{i_q\}_{q=1}^m, \{\delta_j\}_{j \in \{1,\dots,d\} \setminus \{i_q\}_{q=1}^m}\right) \\ &= \{(x_1, \dots, x_d) \in \overline{K}_d \,|\, x_{i_q} \in [0, \alpha_{i_q}], \quad q = 1, \dots, m, \text{ and } x_j = \delta_j \text{ for } j \in \{1, \dots, d\} \setminus \{i_q\}_{q=1}^m\}. \end{aligned}$$

The volume of the sum of all the m-dimensional faces of $\overline{K}_d$ is given by

$$V_m^d := 2^{d-m} \sum_{1 \leqslant i_1 < \cdots < i_m \leqslant d} \prod_{q=1}^{m} \alpha_{i_q}.$$

For example, V_1^d is the perimeter and V_2^d the area of K_d.

Asymptotics for the determinant of the combinatorial Laplacian on graphs have been widely studied, see for example [2,3,5,9,11]. It is related to the number of spanning trees of a graph G, denoted by $\tau(G)$, through the matrix tree theorem due to Kirchhoff (see [1])

$$\tau(G) = \frac{1}{|V(G)|} \det{}^* \Delta_G \tag{1}$$

where $\det^* \Delta_G$ is the product of the non-zero eigenvalues of the Laplacian on G and $|V(G)|$ the number of vertices in G. In [2], the authors developed a technique to compute the asymptotic behaviour of spectral determinants of the combinatorial Laplacian associated to a sequence of discrete tori. The technique consists in studying the asymptotic behaviour of the associated theta function which contains the spectral information of the graph. Consider a graph G with vertex set $V(G)$. For a function f defined on $V(G)$, the combinatorial Laplacian is defined by

$$\Delta_G f(x) = \sum_{y \sim x} (f(x) - f(y))$$

where the sum is over all vertices adjacent to x. Let $\{\lambda_k\}_k$ denote the spectrum of the Laplacian on G. The associated theta function is defined by

$$\theta_G(t) = \sum_{k \in V(G)} e^{-\lambda_k t}.$$

To compute the asymptotic behaviour of spectral determinants on a sequence of d-orthotope square lattices, we express the associated theta function in terms of the theta function associated to the discrete torus of twice the size. This can be done because of the similarity of their spectrum. We adapt the method of the proof of [2, Theorem 3.6] to our case and use the asymptotic results on the torus. The formula obtained relates the determinant of the Laplacian on the discrete lattice $L(n_1, \dots, n_d)$ to the regularized determinant of the Laplacian on the rescaled limiting object, which is the real d-dimensional orthotope K_d, and to the ones on the m-dimensional boundary faces of K_d, $m = 1, \dots, d-1$. Moreover, we compute asymptotic results for the number of rooted spanning forests with 2 and 3 components.

The main result of the paper is stated in the theorem below.

Theorem 1.1. *Given positive integers α_i, $i = 1, \dots, d$, let $\det^* \Delta_{L(\alpha_1 n, \dots, \alpha_d n)}$ be the product of the non-zero eigenvalues of the Laplacian on the d-dimensional orthotope square lattice $L(\alpha_1 n, \dots, \alpha_d n)$. Then*