Applied Mathematics and Computation 324 (2018) 174–190

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Reduced difference polynomials and self-intersection computations

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ARTICLE INFO

Keywords:
Bernstein basis
Polynomial division
Variation-diminishing property
Quadtree decomposition
Self-intersections
Offset curve trimming

ABSTRACT

A *reduced difference polynomial* $f(u,v) = (p(u) - p(v))/(u - v)$ may be associated with any given univariate polynomial $p(t)$, $t \in [0, 1]$ such that the locus $f(u,v) = 0$ identifies the pairs of *distinct* values u and v satisfying $p(u) = p(v)$. The Bernstein coefficients of $f(u,v)$ on $[0, 1]^2$ can be determined from those of $p(t)$ through a simple algorithm, and can be restricted to any subdomain of $[0, 1]^2$ by standard subdivision methods. By constructing the reduced difference polynomials $f(u,v)$ and $g(u,v)$ associated with the coordinate components of a polynomial curve $\mathbf{r}(t) = (x(t), y(t))$, a quadtree decomposition of $[0, 1]^2$ guided by the variation-diminishing property yields a numerically stable scheme for isolating real solutions of the system $f(u,v) = g(u,v) = 0$, which identify self-intersections of the curve $\mathbf{r}(t)$. Through the Kantorovich theorem for guaranteed convergence of Newton–Raphson iterations to a unique solution, the self-intersections can be efficiently computed to machine precision. By generalizing the reduced difference polynomial to encompass products of univariate polynomials, the method can be readily extended to compute the self-intersections of rational curves, and of the rational offsets to Pythagorean–hodograph curves.

1. Introduction

The Bernstein form of a polynomial over a finite domain has found widespread application in diverse contexts, on account of its numerical stability and the intuitive and versatile algorithms it entails [5]. Methods for generalizing the Bernstein form, and for deriving new properties and algorithms, continue to be active areas of investigation [14,21]. In the present study, we investigate the use of the Bernstein form in developing robust algorithms to address the problem of computing the self-intersections of planar curves.

With any univariate polynomial $p(t)$, we may associate a bivariate *reduced difference polynomial* $f(u,v) = (p(u) - p(v))/(u - v)$, such that the points of the algebraic curve $f(u,v) = 0$ identify the pairs of *distinct* values u and v of the independent variable t that satisfy $p(u) = p(v)$. When $p(t)$ is specified in Bernstein form on $t \in [0, 1]$, the Bernstein coefficients of the tensor-product form of $f(u,v)$ on $(u,v) \in [0,1] \times [0,1]$ can be easily obtained from those of $p(t)$, and can be specialized to any rectangular subdomain $[u_1, u_2] \times [v_1, v_2]$ by standard subdivision algorithms. These ideas also generalize to a product $p_1(t)p_2(t)$ of polynomials, in which case the reduced difference polynomial is defined as $f(u,v) = (p_1(u)\, p_2(v) - p_1(v)\, p_2(u))/(u - v)$.

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https://doi.org/10.1016/j.amc.2017.12.016
0096-3003/

The construction of reduced difference polynomials, in conjunction with the subdivision and variation-diminishing properties of the Bernstein form, offers an efficient and extremely robust means of isolating and computing the self-intersections of planar polynomial and rational curves, and also of the offsets to planar Pythagorean–hodograph (PH) curves, which are essential to the offset trimming process. All these problems can be reduced to computing the intersections on $(u,v) \in [0,1] \times [0,1]$ of two algebraic curves $f(u,v)=0$ and $g(u,v)=0$, specified by two reduced difference polynomials.

The intersection points of $f(u,v)=0$ and $g(u,v)=0$ can be isolated by use of a quadtree decomposition of the domain $[\,0,\,1\,] \times [\,0,\,1\,]$ guided by both of these curves. Invoking the variation-diminishing property of the Bernstein form, any subdomain $[u_1,u_2] \times [v_1,v_2]$ on which the coefficients of $f(u,v)$ or $g(u,v)$ are all of the same sign can be discarded as not containing a portion of both curves. Since the quadtree decomposition is governed by *both* curves, it rapidly converges on a set of small subdomains that (potentially) enclose intersection points of $f(u,v)=0$ and $g(u,v)=0$, and an efficient test for the guaranteed convergence of Newton–Raphson iterations to a unique solution within such subdomains allows their rapid computation to machine precision, once they are sufficiently isolated. Since the quadtree subdivision procedure incurs taking only convex combinations of the original Bernstein coefficients of $f(u,v)$ and $g(u,v)$, the method is numerically stable and robust.

Instead of attempting a "synthetic division" of $p(u)-p(v)$ by $u-v$, we formulate a simple recursive algorithm to determine the Bernstein coefficients of $f(u,v)$ from those of $p(t)$. Organizing these coefficients into a matrix, the "boundary" elements are first populated through simple expressions, and the "interior" elements can then be recursively filled in, row-by-row. The process can be further simplified by noting that, since $f(u,v)=f(v,u)$, this matrix is symmetric. It should be noted that the self-intersection algorithms described herein can also be readily adapted to computing the mutual intersections of distinct curves, by considering the bivariate form $p_1(u)-p_2(v)$ for *different* polynomials $p_1(t)$ and $p_2(t)$, and omitting the division by $u-v$.

The remainder of this paper is organized as follows. Section 2 describes the construction of the bivariate difference polynomial associated with a given univariate polynomial. Section 3 then shows how its Bernstein coefficients on any given subdomain $[u_1,u_2] \times [v_1,v_2] \subset [0,1] \times [0,1]$ can be obtained by matrix multiplications. These results are used in Section 4 to generate a quadtree decomposition of $(u,v) \in [0,1] \times [0,1]$ to a prescribed resolution, governed by reduced difference polynomials $f(u,v)=0$ and $g(u,v)=0$ that characterize the self-intersections of planar polynomial curves. A generalized form of the reduced difference polynomial is introduced in Sections 5 and 6, and applied to computing the self-intersections of rational curves, and of the offsets to Pythagorean–hodograph curves. Finally, Section 7 summarizes the methodology proposed herein and suggests possible further developments. The Kantorovich theorem guaranteeing convergence of the Newton–Raphson iteration to a unique solution of the system $f(u,v)=g(u,v)=0$, within a given subdomain, is briefly discussed in an Appendix.

2. Reduced difference polynomials

For any specified univariate polynomial $p(t)$, a bivariate *difference polynomial* $q(u,v) := p(u)-p(v)$ may be defined, whose value is the difference between $p(t)$ at $t=u$ and $t=v$. Since $q(u,v)$ obviously vanishes when $u=v$, it must contain the factor $u-v$. Therefore, $f(u,v) := q(u,v)/(u-v)$ is a polynomial of lower degree, that does not vanish when $u=v$, and the locus $f(u,v)=0$ identifies pairs of *distinct* (u,v) values such that $p(u)=p(v)$. We call $f(u,v)$ the *reduced* difference polynomial associated with $p(t)$.

In dealing with polynomials on finite domains, it is desirable to employ the Bernstein representation, on account of its numerical stability and the many advantageous properties and useful algorithms it entails [5]. The Bernstein basis of degree n on the domain $t \in [\,0,\,1\,]$ is defined by

$$b_i^n(t) := \binom{n}{i}(1-t)^{n-i}t^i \quad i=0,\ldots,n, \tag{1}$$

and a degree-n polynomial $p(t)$ is specified by Bernstein coefficients $a_0,\ldots,a_n$ as

$$p(t) = \sum_{i=0}^{n} a_i\, b_i^n(t).$$

The basis (1) satisfies the partition-of-unity property

$$\sum_{i=0}^{n} b_i^n(t) \equiv 1, \tag{2}$$

which can be used to write $q(u,v) := p(u)-p(v)$ in the tensor-product form

$$q(u,v) = \sum_{j=0}^{n} a_j\, b_j^n(u) - \sum_{k=0}^{n} a_k\, b_k^n(v) = \sum_{j=0}^{n}\sum_{k=0}^{n}(a_j-a_k)\, b_j^n(u)\, b_k^n(v). \tag{3}$$

Since $q(u,v)$ contains the factor $u-v$, dividing it out yields a polynomial of the form