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Pricing and Lot-Sizing for Continuously Decaying Items with Stochastic Demand

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Abstract. The paper is concerned with a stochastic inventory models for continuously deteriorating items with price dependent demand's intensity, zero ending inventory, and non-zero lead time**.** We assume that demand process is a compound Poisson with continuous increments or is described by a Brownian motion process. The objective of this paper is to determine the selling price and lot-size maximizing the average profit per unit time for a lot size large enough**.** We prove that the main part of the mean cycle time as lot-size tends to infinity is the same as for deterministic demand. We obtain the equations for optimal lot-size and time varying selling price.

Keywords: Deterioration, Pricing, Lot-Sizing, Diffusion Process, Compound Poisson, Zero Ending Inventory.

1. INTRODUCTION AND PROBLEM STATEMENT

There are a lot of inventory models for deteriorating items; see, e.g. reviews by Raafat et al. (1991), Goyal and Giri (2001), Bakker et al. (2012). The most recent review is written by Janssen et al. (2016).

A deteriorating inventory model with price-dependent demand was first investigated by Eilon and Mallaya (1966). In order to match real world situations, price-setting inventory models with deterioration are considered in various settings in recent years; see for details Janssen et al. (2016).

According to the reviews, deterministic demand models have received considerably more attention. Stochastic models need more sophisticated methods to analyze and usually it is possible to receive closed form analytical results only by using some approximate methods.

Let us consider the problem statement and introduce some notations. The vendor purchases a quantity Q_0 of instantaneously deteriorating product at a fixed price per unit d (wholesale price) and sells it at a price per unit of the product $c > d$ (retail price) until the product is at hand (zero ending inventory). Then he spends time T_b to order the next lot. For brevity inventory holding cost is not considered. The demand is assumed to be stochastic with price dependent intensity of customers' arrivals. We consider both a compound Poisson and a diffusion demand. The objective is to maximize the average profit per unit time.

Let $Q(t)$ denotes the level of inventory at time t, $Q(0) = Q_0$. Throughout the paper we suppose that the product deteriorates at a constant rate per unit κ, i.e. the changing of $Q(\cdot)$ over time due to the deterioration can be described as $\frac{dQ(t)}{dt} = -\kappa Q(t)$.

2. MEAN CYCLE TIME FOR DETEORATING ITEMS

2.1 Deterministic demand

We initially address a simple deterministic model. Let the intensity of the customers' flow $\lambda(c)$ depends on the retail price and each of the customers purchases the same value a_1. Then function $Q(\cdot)$, $Q(t) > 0$, is described by the following equation:

$$\frac{dQ(t)}{dt} = -\kappa Q(t) - a_1 \lambda(c) . \quad (1)$$

The model also known as exponential decay inventory model; see, e.g. Nahmias (1982). Ghare and Schrader (1963) were the first who developed the model. Cohen (1977) was one of the first to consider pricing and inventory decision for an exponentially decaying inventory under deterministic demand.

From (1) we receive the well-known formula (see, e.g. Cohen, 1977)

$$Q(t) = Q_0 e^{-\kappa t} - \frac{a_1 \lambda(c)}{\kappa}\left(1 - e^{-\kappa t}\right).$$

Solving the equation

$$Q_0 e^{-\kappa T_0} - \frac{a_1 \lambda(c)}{\kappa}\left(1 - e^{-\kappa T_0}\right) = 0$$

with respect to T_0, we find the length of time it takes to sell lot size Q_0

$$T_0 = \frac{1}{\kappa} \ln\left(1 + \frac{Q_0 \kappa}{a_1 \lambda(c)}\right).$$

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So the corresponding profit

$$\Pi = a_1 c \lambda(c) T_0 - d \cdot Q_0 ,$$

and the profit per unit time

$$\mathrm{P} = \frac{a_1 c \lambda(c) \cdot \ln\left(1 + Q_0 \kappa / a_1 \lambda(c)\right) - d \cdot \kappa Q_0}{\ln\left(1 + Q_0 \kappa / a_1 \lambda(c)\right) + \kappa T_b} =$$
$$= a_1 c \lambda(c) \cdot \frac{\ln(1+z) - (d/c) z}{\ln(1+z) + \kappa T_b}, \tag{2}$$

where dimensionless variable $z = \kappa Q_0 / a_1 \lambda(c)$.

2.2 Compound Poisson demand with exponential batch size distribution

Let the demand be a Poisson process with constant intensity λ , and the values of purchases (batch sizes) are i.i.d. exponentially distributed random variables with mean a_1. Let $T(Q)$ be a mean value of T_0 given $Q_0 = Q$.

Let Δt be a short interval of time. Then we have the following equation for $T(\cdot)$:

$$T(Q) = \Delta t + (1 - \lambda \Delta t) T(Q - \kappa Q \Delta t) +$$
$$+ \lambda \Delta t \int_0^Q T(Q - x) p(x) dx + o(\Delta t) = \Delta t +$$
$$+ (1 - \lambda \Delta t)\left(T(Q) - \kappa Q \Delta t \frac{dT}{dQ} \right) + \lambda \Delta t \int_0^Q T(Q - x) p(x) dx + o(\Delta t),$$

where $p(\cdot)$ is the probability density function of batch sizes. Dividing by Δt and letting $\Delta t \to 0$ give us the following integro-differential equation:

$$T'(Q)\kappa Q + \lambda T(Q) - \lambda \int_0^Q T(Q - x) p(x) dx = 1 . \tag{3}$$

For exponentially distributed purchases we can write (3) as

$$(T'(Q)\kappa Q + \lambda T(Q)) \exp\left(Q / a_1\right) -$$
$$- \frac{\lambda}{a_1} \int_0^Q T(z) \exp\left(x / a_1\right) dx = \exp\left(Q / a_1\right) .$$

Taking the derivative we receive differential equation

$$\kappa Q T''(Q) + \left(\kappa + \lambda + \frac{\kappa Q}{a_1} \right) T'(Q) = \frac{1}{a_1} . \tag{4}$$

The exact solution of (4) has rather complicated form. So let us find the asymptotic solution for Q large enough. We will find it in such a form:

$$T(Q) = A \ln(1 + \beta Q) , \tag{5}$$

where A and β are unknown constants.

Substituting (5) into (4) we get

$$- \frac{A \beta^2 \kappa Q}{(1 + \beta Q)^2} + \frac{A \beta \left(\kappa + \lambda + \kappa Q / a_1 \right)}{1 + \beta Q} = \frac{1}{a_1} . \tag{6}$$

As $Q \to \infty$, it follows $A = 1/\kappa$.

Substituting A into (6) we get

$$- \frac{\beta^2}{(1 + \beta Q)^2} + \frac{1}{1 + \beta Q} \left(\beta \left(1 + \frac{\lambda}{\kappa} \right) - \frac{1}{a_1} \right) = 0 .$$

Multiply both sides of the equation by $1 + \beta Q$ and then tend Q to infinity. It follows $\beta = \kappa / a_1 \lambda$, i.e. for the mean cycle time we have the same result for $Q >> 1$ as in 2.1 point.

2.3 Compound Poisson demand with any batch size distribution

The result can be generalized to any batch size distribution $p(\cdot)$ such that integral $\int_Q^\infty p(x) dx$ tends to zero at least as fast as $\exp(-Q)$ as $Q \to +\infty$.

Using the same procedure and substituting (5) into (3), we get

$$\frac{A \kappa \beta Q}{1 + \beta Q} + \lambda A \ln(1 + \beta Q) - \lambda A \int_0^Q \ln(1 + \beta(Q - x)) p(x) dx = 1 . \tag{7}$$

Taking into account the asymptotic condition we can rewrite (7) for Q large enough in the following equivalent form:

$$\frac{A \kappa \beta Q}{1 + \beta Q} + \lambda A \ln(1 + \beta Q) \int_0^Q p(x) dx -$$
$$- \lambda A \int_0^Q \ln(1 + \beta(Q - x)) p(x) dx = 1$$

or