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Improved exact algorithms to economic lot-sizing with piecewise linear production costs

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ABSTRACT

In this article we study a classical single-item economic lot-sizing problem, where production cost functions are assumed to be piecewise linear. The lot-sizing problem is fundamental in lot-sizing research, and it is applicable to a wide range of production planning models. The intractability of the problem is related to the value of m, the number of different breakpoints of the production cost functions. When m is arbitrary, the problem is known to be NP-hard (Florian, Lenstra & Rinnooy Kan, 1980). However, if m is fixed, then it can be solved in polynomial time (Koca, Yaman & Akturk, 2014). So far, the most efficient algorithm to solve the problem has a complexity of $O(T^{2m+3})$, where T is the number of periods in the planning horizon (Koca et al., 2014). In this article we present an improved exact algorithm for solving the problem, where the complexity is reduced to $O(T^{m+2} \cdot \log T)$. As such it also improves upon the computational efficiency of solving some existing lot-sizing problems which are important special cases of our model. In order to achieve a more efficient implementation, our algorithm makes use of a specific data structure, named range minimum query (RMQ).

1. Introduction

The single-item economic lot sizing problem that we study can be described as follows: There are T time periods in the planning horizon, and the demand of period j is known to be d_j, $j = 1, 2, \ldots, T$. Let X_j denote the production quantity in period j, and let I_j denote the amount of inventory at the end of period j. Backlogging is allowed. In each period j the inventory cost function H_j is assumed to be general concave over intervals $(-\infty, 0]$ and $[0, +\infty)$, respectively, with $H_j(0) = 0$. Let P_j denote the production cost function in period j. We assume P_j to be a piecewise linear function with m different but stationary breakpoints, where m is assumed to be a small constant. Let $B_1, B_2, \ldots, B_m$ denote the m breakpoints. For notational convenience, we define $B_0 = 0$ and $B_{m+1} = \sum_{j=1}^{T} d_j$, and assume $B_0 < B_1 < \cdots < B_m < B_{m+1}$. Function P_j is assumed to be linear over interval $(B_{\ell-1}, B_\ell]$ for $\ell = 1, 2, \ldots, m+1$. In particular, we assume that $P_j(0) = 0$ and

$$P_j(X_j) = s_{j,\ell} + p_{j,\ell} \cdot X_j$$

if $B_{\ell-1} < X_j \le B_\ell$ for $\ell = 1, 2, \ldots, m+1$. Here $s_{j,\ell}$ and $p_{j,\ell}$ represents the corresponding fixed setup cost and variable unit production cost, respectively, when X_j (the production quantity in period j) is within interval $(B_{\ell-1}, B_\ell]$. The objective is to minimize the total production and inventory cost to satisfy the demand of the T periods. Define $I_0 = 0$. Our lot-sizing problem can be formulated as the following mathematical program:

$$\text{minimize} \quad \sum_{j=1}^{T} \Big[P_j(X_j) + H_j(I_j) \Big] \tag{1}$$

$$\text{subject to} \quad I_j = I_{j-1} + X_j - d_j \quad (j = 1, 2, \ldots, T), \tag{2}$$

$$I_0 = I_T = 0, \tag{3}$$

$$X_j \ge 0 \quad (j = 1, 2, \ldots, T). \tag{4}$$

Objective function (1) includes the total production and inventory cost. Constraint (2) is the inventory balance equation. Constraint (3) states the initial and terminal conditions. Constraint (4) requires X_j to be non-negative. We call this problem the "economic lot-sizing problem with piecewise linear production costs" and abbreviate it with ELS-PL.

In each period the production cost function of ELS-PL is piecewise linear. One motivation of piecewise linear production costs is as follows. Consider the scenario in which a manufacturer has k decentralized production centers to make a single product to meet the deterministic demands over T time periods. Let $\mathbf{PC}_i$ denote the ith production center, $i = 1, 2, \ldots, k$. The capacity of $\mathbf{PC}_i$

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is c_i. For any period j and any production quantity $X \in (0, c_i]$, the cost of producing X units of product in $\mathbf{PC}_i$ is equal to $\alpha_{i,j} + p_{i,j} \cdot X$, where $\alpha_{i,j}$ and $p_{i,j}$ are the corresponding fixed setup cost and variable unit production cost, respectively. Let $\beta_{i,j} = (\alpha_{i,j} + p_{i,j} \cdot c_i)/c_i$ denote the average unit production cost of $\mathbf{PC}_i$ in period j when $\mathbf{PC}_i$ is fully used. Without loss of generality, we assume $\beta_{1,j} \le \beta_{2,j} \le \cdots \le \beta_{k,j}$. A production center with a smaller index is assumed to have a higher priority to make product. Particularly, in each period $\mathbf{PC}_{i+1}$ is used to make product only if $\mathbf{PC}_i$ has been fully used, $i = 1, 2, \ldots, k-1$. Let $m = k - 1$. Let $s_{\ell,j} = \sum_{i=1}^{\ell-1} [\alpha_{i,j} + (p_{i,j} - p_{\ell,j}) \cdot c_i]$ and $B_\ell = \sum_{r=1}^{\ell} c_r$ for $\ell = 1, 2, \ldots, m+1$. For any total production quantity $X_j > 0$ in period j, if $X_j \in (B_{\ell-1}, B_\ell]$ for some $\ell \in \{1, 2, \ldots, k\}$, then the total production cost in the period is equal to

$$\sum_{i=1}^{\ell-1} (\alpha_{i,j} + p_{i,j} \cdot c_i) + p_{\ell,j} \cdot (X_j - \sum_{i=1}^{\ell-1} c_i)$$
$$= \sum_{i=1}^{\ell-1} [\alpha_{i,j} + (p_{i,j} - p_{\ell,j}) \cdot c_i] + p_{\ell,j} \cdot X_j = s_{\ell,j} + p_{\ell,j} \cdot X_j = P_j(X_j),$$

which is exactly the same as the piecewise linear production cost in period j of ELS-PL.

In the economic lot-sizing (ELS) literature, Koca, Yaman, and Akturk (2014) studied the ELS problem with piecewise concave production costs, where the production cost function in each period has a fixed number of m stationary breakpoints. They presented an $O(T^{2m+3})$ exact algorithm. Our lot-sizing problem ELS-PL is a special case of their model. To the best of our knowledge, the $O(T^{2m+3})$ algorithm in Koca et al. (2014) is also the most efficient exact algorithm to solve ELS-PL. The aim of this article is to develop a faster exact algorithm to solve ELS-PL. The complexity of our algorithm is $O(T^{m+2} \cdot \log T)$. We makes use of the piecewise linear structure of the production cost functions to design our algorithm. The major contribution of this article is the following main theorem:

Theorem 1. *Problem ELS-PL can be solved in $O(T^{m+2} \log T)$ time.*

ELS-PL is a fundamental problem in inventory and production planning research. Numerous existing economic lot-sizing (ELS) problems are special cases of ELS-PL. For examples, ELS with subcontracting (Lee & Li, 2013; Lee & Zipkin, 1989), ELS with quantity discounts (Chan, Muriel, Shen, & Simchi-Levi, 2002; Li, Ou, & Hsu, 2012), ELS with multiple transportation modes (Jaruphongsa, Çetinkaya, & Lee, 2005), ELS with minimum ordering quantity (Hellion, Mangione, & Penz, 2012, 2013), among others, are famous special cases of ELS-PL. Hence, the solution method proposed in this article is applicable to solve a wide range of ELS problems. We would like to point out that our algorithm is applicable to improve the computational efficiency of a number of famous ELS problems in the literature. For example, Lee and Li (2013) studied an ELS model with capacity reservation, where the production cost functions were piecewise linear with one stationary breakpoint. They presented an $O(T^4)$ optimal algorithm. It only takes $O(T^3 \log T)$ time by applying our algorithm. For another example, Hellion et al. (2012, 2013) presented an $O(T^6)$ optimal algorithm for an ELS model with minimum ordering quantity, where the production cost functions were piecewise concave with two stationary breakpoints. By applying our algorithm, it only takes $O(T^4 \log T)$ time when the production cost functions are piecewise linear instead of piecewise concave. Li et al. (2012) studied the lot-sizing problem with all units discount and resales under the assumptions that the breakpoints of the cost functions are stationary, and backlogging is not allowed. They developed an $O(T^{\mu+3})$ time exact algorithm, where μ is the number of stationary breakpoints. Their algorithm can be extended to solve the more general case when backlogging is allowed, but the time complexity is increased to $O(T^{2\mu+3})$. By applying our algorithm, it only takes $O(T^{\mu+3} \log T)$ time.

To help the readers easier follow our algorithms, we provide a small example E1 that will be used throughout the article. Example E1 is as follows: $T = 6$, $(d_1, d_2, d_3, d_4, d_5, d_6) = (4, 4, 1, 8, 5, 4)$; $m = 1$, $B_1 = 7$; for $j = 1, \ldots, 6$, $P_j(x) = 1 + 2x$ if $0 < x \le 7$, and $P_j(x) = 6 + x$ if $x > 7$; for $j = 1, \ldots, 6$, $H_j(y) = y$ if $y \ge 0$, and $H_j(y) = -2y$ if $y < 0$.

The rest of the article is organized as follows. First, a literature review is presented in Section 2. Second, an $O(T^{2m+3})$ exact algorithm is developed in Section 3. Third, we show how to reduce the time complexity to $O(T^{m+2} \cdot \log T)$ in Section 4. Finally, some conclusion remarks are provided in Section 5.

2. Literature review

Since Wagner and Whitin wrote their seminal paper (Wagner & Whitin, 1958), the basic ELS problem and some of its important variants have received extensive attentions. ELS-PL is an important variant of the basic ELS problem because the production cost functions of ELS-PL have a very general structure. Many variants of the basic ELS model can be transferred into special cases of ELS-PL. Hence, to improve the computational efficiency of solving ELS-PL is important in ELS research.

ELS problems with general cost functions have been studied by early lot-sizing researchers. Florian and Klein (1971) provided an optimal algorithm with complexity of $O(\hat{D} \cdot \hat{C})$ for the case with the most general production cost structure, where $\hat{D}$ is the total demand and $\hat{C}$ is the total production capacity in the planning horizon, respectively. Shaw and Wagelmans (1998) developed an $O(T \cdot \bar{m} \cdot \hat{D})$ exact algorithm for case with piecewise linear production costs and general inventory holding costs, where $\bar{m}$ is the average number of breakpoints of the production cost functions. Love (1973), Swoveland (1975), Chen, Hearn, and Lee (1994a, 1994b), Baker, Dixon, Magazine, and Silver (1978) and Leung, Magnanti, and Vachani (1989), among others, studied ELS models with piecewise concave/linear production/inventory costs. But all of those optimal algorithms mentioned above were either pseudo-polynomial or exponential.

The capacitated economic lot-sizing (CELS) problem is known to be NP-hard in general (see Florian, Lenstra, & Kan, 1980). In CELS, the production capacity of each period is different, which indicates that each production cost function only has one different breakpoint. It is also well-known that the economic lot-sizing problem with constant capacity and concave production/inventory costs (ELS-CC) can be solved in polynomial time (see Florian et al., 1980; Van Hoesel, & Wagelmans, 1996, and Ou, 2012). In ELS-CC, the production capacity of each period is stationary, which indicates that all production cost functions have only one stationary breakpoint. It turns out that an ELS problem is polynomially solvable if the production costs are piecewise concave but the number of breakpoints is fixed (Koca et al., 2014).

Chan et al. (2002) considered a lot-sizing model with a modified all-units discount scheme, where the production cost is assumed to be piecewise linear. They showed that their lot-sizing problem remains to be NP-hard even when the production cost functions are stationary if the number of price breakpoints in each production cost function is arbitrary. Their paper gave an open question of the complexity of their lot-sizing problem when the production cost functions are stationary and the number of price breakpoints is fixed. This open question was answered by Li et al. (2012). Li et al. (2012) studied ELS with all-units discount and resales. They showed that their lot-sizing problem could be solved in $O(T^{\mu+3})$ time if μ, the number of different price breakpoints in the production cost functions, is fixed. They also presented an

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