ARTICLE IN PRESS

Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Pure-strategy Nash equilibria on competitive diffusion games

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A R T I C L E   I N F O

Article history:
Received 20 July 2015
Received in revised form 28 November 2017
Accepted 7 March 2018
Available online xxxx

Keywords:
Games on graphs
Pure Nash equilibrium
Discrete Voronoi game
Information diffusion game

A B S T R A C T

This paper treats two types of competitive facility location games on graphs: information diffusion games and discrete Voronoi games. Both of these games can be regarded as models of the rumor spreading processes on the networks, where each player of the game wants to select an influencer who can widely spread information throughout the network. For each game, given a graph and the number of players, we are interested in whether there exist pure Nash equilibria or not. In this paper, we discuss the existence of pure Nash equilibria on graphs with small diameter, path graphs, and cycle graphs. The results include the behavior of the discrete Voronoi games on graphs with diameter two, and the complete characterization of the existence of the pure Nash equilibria in the discrete Voronoi games and information diffusion games on path graphs.

1. Introduction

Competitive facility location games are the models of rivaling sellers seeking positions to maximize their market share. A foundation of competitive facility location games is known as the ice-cream vendor problem by Hotelling [5]. Recently, competitive facility location games on graphs are widely studied [2,4,11]. Such games on graphs can model, for example, the process of rumors about some products spreading through social networks. Each competitive firm wants to select several influencers from whom information diffuse throughout the network efficiently.

In this paper, we focus on two kinds of competitive facility location games. One is the *information diffusion game*, which is introduced by Alon et al. [1], and the other is the *discrete Voronoi game* discussed by Dürr and Tang [3]. In both of these games, each player selects one vertex on a given graph, and sends his/her message to this selected vertex. After that, each informed vertex sends the received message to all its uninformed neighbors in each step. The utility of each player is given by the number of vertices received his/her message when all the diffusion process is finished. The difference of the two models is the behavior of the vertices which receive more than one message at the same time. In the former model, messages vanish due to collisions of messages, and the diffusion process is prevented at the vertices where collisions occur. In the latter model, on the other hand, when more than one message arrives to one vertex at the same time, the vertex receives all of them and the utility is shared equally. Thus, all vertices contribute to the utility of some players. Note that the game proposed by [10] is different from our discrete Voronoi game although it uses the same name.

We investigate the existence of Nash equilibria for our games. Nash equilibria are the stable states of the game in which no player can improve his/her utility by unilaterally changing to a different strategy. Determining whether Nash equilibria exist and effectively computing them have attracted many researchers in economics, decision making and computer science.

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https://doi.org/10.1016/j.dam.2018.03.031

Our interest is on the existence of Nash equilibria in the case of pure strategies, where each player chooses an action deterministically. We do not deal with mixed strategies in this paper. Nash equilibria for pure strategies are briefly referred to as *pure Nash equilibria*. The existences of pure Nash equilibria for information diffusion games and discrete Voronoi games are studied by [1,3]. Dürr and Tang [3] showed a relatively simple graph that does not allow a pure Nash equilibrium for discrete Voronoi games even though there are only two players.

In this paper we characterize the existence of pure Nash equilibria on graphs with small diameter, path graphs, and cycle graphs. After giving formal definitions of information diffusion games and discrete Voronoi games in Section 2, we discuss the existence of the pure Nash equilibria for graphs with small diameters in Section 3. In Section 4, we show the complete characterization of the existence of the pure Nash equilibria in the discrete Voronoi games and information diffusion games on path graphs. We add some observations for the discrete Voronoi games on trees in Section 5.

2. Information diffusion game and discrete Voronoi game

Let $G = (V, E)$ be an undirected graph and N a set of players. Throughout this paper we always assume G is connected. Suppose that the graph G has n vertices and there are k players, that is, let us assume $|V| = n$ and $|N| = k$. In both of the two games, information diffusion games and discrete Voronoi games, the strategy set of each player is V and the strategy profile of players is a vector $\boldsymbol{x} = (x_1, x_2, \dots, x_k) \in V^N$ associating each player i to the vertex x_i selected initially. By $\ell(\boldsymbol{x})$, we denote the number of vertices selected by players initially, that is, $\ell(\boldsymbol{x}) = |\{x_i | i \in N\}|$. If there is no confusion we write ℓ instead of $\ell(\boldsymbol{x})$.

In the information diffusion game on G, each player wants to spread his/her own message to as many vertices in the graph G as possible. During the process of spreading the messages, each vertex admits three states: *uninformed*, *informed*, and *deadlocked*. Initially, every vertex is uninformed. At time one, each player i selects one vertex x_i from V and sends his/her message to the selected vertex. The choice of each player in this first step is represented by the strategy profile $\boldsymbol{x} = (x_1, x_2, \dots, x_k)$. Then, a vertex that is sent only one message, i.e., a vertex selected by exactly one player, receives the message and turns into the state of informed, while a vertex that is sent more than one message becomes deadlocked. At time $t + 1$, each informed vertex that received a message at time t sends the same message to all of its adjacent uninformed vertices, and these messages are processed in the same way: an uninformed vertex that is sent only one message receives the message and turns into the state of informed, while an uninformed vertex that is sent more than one message becomes deadlocked. Deadlocked vertices do not receive any messages. This diffusion process finishes when there is no vertex that newly becomes informed. Given a strategy profile $\boldsymbol{x}$ which stands for a vertex selected by each player at time one, the utility of player i, denoted by $U_i(\boldsymbol{x})$, is given by the number of informed vertices that receive the message of player i at the end of the diffusion process.

In the discrete Voronoi game, we use a notation $C_v(\boldsymbol{x})$, for a vertex v and a strategy profile $\boldsymbol{x} = (x_1, \dots, x_k)$, to represent the set of players i whose selected vertex x_i is closest to v among $x_1, \dots, x_k$. Namely, $C_v(\boldsymbol{x}) = \arg\min\{d(x_i, v) \mid i \in N\}$, where $d(u, v)$ denotes the shortest path length between $u \in V$ and $v \in V$ in G. The utility $\tilde{U}_i(\boldsymbol{x})$ of player i in the discrete Voronoi game is given by $\sum\{\frac{1}{|C_v(\boldsymbol{x})|} \mid v \in V, i \in C_v(\boldsymbol{x})\}$.

The information diffusion game and the discrete Voronoi game are similar in the sense that every vertex is assigned to the closest players and utility is given by the number of vertices assigned to each player. The difference of these two games happens when a vertex receives more than one message simultaneously. Such vertices will be shared by the players and the messages will be propagated further in the discrete Voronoi game, while they remain just in the deadlocked state and the propagation of the messages stops at them in the information diffusion game. Thus, for the initial location, a player may improve his/her utility by colocating with some other player in the Voronoi game. On the other hand the player gets no utility by colocating in the information diffusion game. Especially, in the discrete Voronoi game we always have $\sum_{i\in N} \tilde{U}_i(\boldsymbol{x}) = |V|$ for any $\boldsymbol{x}$, which does not necessarily hold in the information diffusion game.

Given a strategy profile $\boldsymbol{x} = (x_1, \dots, x_k)$ and $x' \in V$, we denote by $(x', \boldsymbol{x}_{-i})$ the vector equal to $\boldsymbol{x}$ but with the ith component replaced by x', that is, $(x', \boldsymbol{x}_{-i}) = (x_1, \dots, x_{i-1}, x', x_{i+1}, \dots, x_k)$. A strategy profile $\boldsymbol{x}$ is a pure Nash equilibrium of the information diffusion game (resp. the discrete Voronoi game), if $U_i(x', \boldsymbol{x}_{-i}) \le U_i(\boldsymbol{x})$ (resp. $\tilde{U}_i(x', \boldsymbol{x}_{-i}) \le \tilde{U}_i(\boldsymbol{x})$) for any player i and any $x' \in V$.

Throughout this paper, we assume that $k \le n$, otherwise both games have trivial structures. Namely, when $k > n$, a strategy profile is a Nash equilibrium of the information diffusion game if and only if there are no unoccupied vertices, and it is a Nash equilibrium of the discrete Voronoi game if and only if the numbers of players located on the vertices differ by at most one between any two vertices.

3. Existence of pure Nash equilibria on graphs with small diameter

This section discusses the existence of pure Nash equilibria of our games on graphs having small diameter, which was firstly discussed by Alon et al. [1] for information diffusion games. The diameter of a graph is defined by the maximum distance between a pair of vertices, that is, defined by $\max\{d(v, u) \mid u, v \in V\}$. Alon et al. have shown that the information diffusion game on a graph whose diameter is at least three may not possess a pure Nash equilibrium, even for two players. Although their example is given by a large graph, the smaller graph shown in Fig. 1 is also an example on which the information diffusion game does not admit pure Nash equilibria for two players.

Moreover, Takehara et al. [9] have shown the following.

Please cite this article in press as: H. Enomoto, et al., Pure-strategy Nash equilibria on competitive diffusion games, Discrete Applied Mathematics (2018), https://doi.org/10.1016/j.dam.2018.03.031.