Insurance: Mathematics and Economics 49 (2011) 418–428

Contents lists available at SciVerse ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

# Behavioral optimal insurance

K.C.J. Sung [a], S.C.P. Yam [b], S.P. Yung [a], J.H. Zhou [a,*]

[a] Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong
[b] Department of Statistics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong

ARTICLE INFO

*Article history:*
Received March 2010
Received in revised form
April 2011
Accepted 28 April 2011

*Keywords:*
Optimal insurance
Behavioral finance
Cumulative prospect theory
Non-convex optimization
Generalized insurance layer

ABSTRACT

The present work studies the optimal insurance policy offered by an insurer adopting a proportional premium principle to an insured whose decision-making behavior is modeled by Kahneman and Tversky's Cumulative Prospect Theory with convex probability distortions. We show that, under a fixed premium rate, the optimal insurance policy is a generalized insurance layer (that is, either an insurance layer or a stop–loss insurance). This optimal insurance decision problem is resolved by first converting it into three different sub-problems similar to those in Jin and Zhou (2008); however, as we now demand a more regular optimal solution, a completely different approach has been developed to tackle them. When the premium is regarded as a decision variable and there is no risk loading, the optimal indemnity schedule in this form has no deductibles but a cap; further results also suggests that the deductible amount will be reduced if the risk loading is decreased. As a whole, our paper provides a theoretical explanation for the popularity of limited coverage insurance policies in the market as observed by many socio-economists, which serves as a mathematical bridge between behavioral finance and actuarial science.


## 1. Introduction

Insurance is a financial instrument used to protect against the risk of a contingent loss; the insured transfers a portion of it to the insurer in exchange for a premium. For decades, there are substantial theoretical and empirical studies to determine the form of insurance that is most preferred by the insureds. In Expected Utility Theory (first axiomatized by von Neumann and Morgenstern, 1944), in which every rational individual is assumed to be an expected utility maximizer, Arrow (1971, 1974) showed that if the insurer is risk neutral, the optimal insurance policy provides full coverage but not a nondegenerate stop–loss policy. In relation to Pareto optimality, Borch (1975) and Raviv (1979) also obtained similar results and concluded that no insurance policy with upper coverage limit is Pareto optimal if all participating agents are risk-averse. For further extensions with relaxation of constraints on feasible insurances, modification of premium calculating principles, and with additional budget constraints can be found in the works of Blazenko (1985), Cummins and Mahul (2004), Deprez and Gerber (1985), Gollier (1987), Kaluszka (2005), Promislow and Young (2005) and Zhou and Wu (2008). Recently, different risk measures have been proposed in characterizing optimal (re)insurances. For the studies in this direction, see Balbás et al. (2008), Cai and Tan (2007), Cai et al. (2008), Cheung (2010), Cheung et al. (submitted for publication), Kaluszka (2001, 2004).

On contrary to those theoretical results under Expected Utility Theory, the form of optimal insurance resulted in empirical studies is completely different. For example, health economist—Fuchs (1976, 1991) revealed the popularity of insurance with low deductibles and cap which is actually common in medical insurance Fuchs (1976, 1991):

> "…(C)onsumers should prefer major medical (catastrophe) insurance, that is, plans with substantial deductibles or co-payment provisions for moderate expenses but ample coverage for very large expenses. Instead, we observe a strong preference for 'first dollar' or shallow coverage …",

this observation is hitherto puzzling as they are not Pareto-optimal under the classical framework. Besides, Slovic et al. (1977) conducted a series of laboratory studies of insurance decision-making and found that people prefer protection against events that inflict a small loss rather than rare large-losses. They proffer two explanations for this phenomenon, both contrary to the Expected Utility Theory: the insureds are risk-seeking towards loss and they tend to refuse protection against losses whose probability is below some threshold. Kunreuther et al. (1978), Schoemaker and Kunreuther (1979), and Hershey and Schoemaker (1980) also provided experimental evidence for these claims.

In order to better explain the observed insurance buying behavior, a number of non-expected utility models have been proposed including subjectively weighted utility, Choquet expected

* Corresponding author. Tel.: +852 68093831.
*E-mail address:* mrunderhill1980@gmail.com (J.H. Zhou).

0167-6687/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.insmatheco.2011.04.008

utility and Rank-dependent expected utility (Carlier and Dana, 2003, 2007; Karmarkar, 1978; Machina, 2004). However, the proposed frameworks do not have sufficient empirical support; and in some models, it is counter-intuitive that the optimal insurance is discontinuous in the realized loss amount.

In this paper, we shall adopt the framework as first formulated by Jin and Zhou (2008) via Kahneman and Tversky's Cumulative Prospect Theory (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992). Cumulative Prospect Theory is developed, based on a series of laboratory experiments, to model the decision-making behavior of the insured. There are three essential features in this model: (1) people make decisions based on gains and losses (relative to some reference point) in wealth, rather than terminal wealth; (2) the value index function is S-shaped, that is concave for relative gains while convex for relative losses, to show risk aversion for gains and risk seeking for losses; and (3) the existence of probability distortion, that is the perceived probability from its true value, which overweights small probabilities and underweights large probabilities. Due to both the non-convex and non-linear nature of the objective function, rigorous analytical treatment has been absent in this behavioral framework until a recent breakthrough provided by Jin and Zhou (2008).

According to Slovic et al. (1977), people do not use insurance to protect against rare large losses which they tend to neglect; in other words, people underweight all uncertain events, not only 'average', whenever they have the incentive to buy insurance. Mathematically speaking, the buyer has a convex probability distortion, which is also justified as in the work of Yaari (1987). To avoid moral issues or insurance swindles, any feasible insurance policy should satisfy the following assumptions: (1) an additional unit of loss cannot result in more than a unit increment of indemnity claim; (2) indemnity schedule is a continuous function of the claim size; and finally (3) as the buyer expected, for any additional loss claim, at least not lesser compensation that would be requested.

Under the framework of the Cumulative Prospect Theory, in this paper, we shall explain that policies with an upper limit on coverage are indeed Pareto optimal and result from the decision-making preference of the insured (instead of the insurer); we shall consider a maximization problem of a difference of two Choquet-type integrals having convex probability distortions, subject to the proportional premium principle, among all feasible insurance policies satisfying assumptions (1)–(3). Because of the non-convex and non-linear nature, no standard tools in convex optimization can be applied. By converting into three different sub-problems similar to those in Jin and Zhou (2008), the problem can still be resolved; however, as we are asking for a more regular optimal solution, a completely different approach is developed to tackle these sub-problems.

Given a fixed premium rate, we shall show that a generalized insurance layer (that is, either an insurance layer or a stop–loss insurance) is an optimal insurance policy. Furthermore, when the premium rate is regarded as a decision variable and there is no risk loading, the optimal insurance in this form would have no deductibles but with a cap. The result also suggests that the deductible amount will be reduced if the risk loading is decreased. All of these results are in line with the historical observations as mentioned above.

In summary, the main contributions of this paper are: (1) provide a mathematical bridge that connects a well-known theory in behavioral finance (which has sufficient empirical evidence and experience support) to real-life insurance buying behaviors, and hence a quantitative justification and resolution to the socioeconomic enigma as found in Fuchs (1976, 1991), and (2) laying down some key mathematical tools for future quantitative treatment of similar optimal insurance problems in the behavioral framework. The rest of the paper is organized as follows. In Section 2, the optimal insurance design problem under the framework of the Cumulative Prospect Theory is formulated, and it is converted into three sub-problems in Section 3. An optimal policy is obtained in Section 4 by tackling the sub-problems with a completely different method from Jin and Zhou (2008). In Section 5, optimal insurance policy is established when the premium is also a decision variable. In Section 6, we consider the case that the reference point is non-zero. Further numerical examples can be found in Section 7.

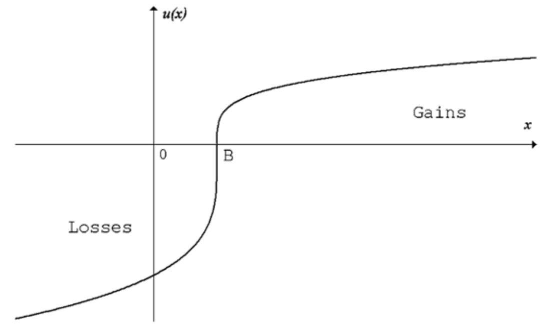


**Fig. 1.** Typical S-shaped utility function.

## 2. Problem formulation

In this paper, an insured is assumed to seek for protection against a potential loss $X$, which is an integrable nonnegative continuous random variable. We denote its density function and cumulative distribution function by $f(x)$ and $F(x)$ respectively. Suppose that the insured has an initial wealth $w > 0$ and let $P$ be the premium of the indemnity schedule $I(\cdot)$. If the loss $X$ occurs, her wealth becomes $I(X) - X + w - P$.

The indemnity schedule $I(\cdot)$ is continuous with $I(0) = 0$ and satisfies

$$0 \leq I(x_2) - I(x_1) \leq x_2 - x_1, \quad \text{for } x_1 \leq x_2. \tag{1}$$

That is, $I(\cdot)$ is non-decreasing and any increment in compensation is always less than or equal to the increment in the amount of loss. Suppose the premium is calculated under the proportional premium principle:

$$(1+\theta)\mathbb{E}[I(X)] = P,$$

where $\theta > 0$.

The decision-making behavior of the insured is modeled by Kahneman and Tversky's Cumulative Prospect Theory (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992), which has the following three essential features:

- The insured makes a decision based on gains and losses relative to a reference point $B$ in wealth, rather than her terminal wealth.
- The S-shaped value index function $u(x)$ is defined by

  $$u(x) = u_+(x-B)^+ - u_-(x-B)^-,$$

  where $u_+(\cdot)$ and $u_-(\cdot)$ are two mappings from $\mathbb{R}^+$ to $\mathbb{R}^+$. We assume that $u_+(\cdot)$ and $u_-(\cdot)$ are continuously differentiable, strictly increasing and strictly concave with $u_+(0) = 0$ and $u_-(0) = 0$, as shown in Fig. 1.
- The insured's probability distortion for gains and losses are represented respectively by two continuously differentiable mappings $g_1$ and $g_2$ from [0, 1] to [0, 1], which are strictly increasing and satisfy $g_1(0) = g_2(0) = 0$, $g_1(1) = g_2(1) = 1$. To reflect the incentive of the insured, we assume that $g_1$ and $g_2$ are both convex functions, as shown in Fig. 2.

The objective in this paper is to find an indemnity schedule $I(\cdot)$ which maximizes the insured's value of her terminal wealth. As in Jin and Zhou (2008), the buyer's value is formulated as $V(I, P, B) \triangleq V_+(I, P, B) - V_-(I, P, B)$, where