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An analytic method for sensitivity analysis of complex systems

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HIGHLIGHTS

- An analytic formula for the determination of output uncertainty is deduced.
- The general approximation for output variance is compared to the exact value.
- Higher-order impacts from $V(x)$ cannot be neglected for highly non-linear models.
- The importance of each input is quantified for two complex systems.

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ABSTRACT

Sensitivity analysis is concerned with understanding how the model output depends on uncertainties (variances) in inputs and identifying which inputs are important in contributing to the prediction imprecision. Uncertainty determination in output is the most crucial step in sensitivity analysis. In the present paper, an analytic expression, which can exactly evaluate the uncertainty in output as a function of the output's derivatives and inputs' central moments, is firstly deduced for general multivariate models with given relationship between output and inputs in terms of Taylor series expansion. A γ-order relative uncertainty for output, denoted by $\mathrm{R}_{\mathrm{V}}^{\gamma}$, is introduced to quantify the contributions of input uncertainty of different orders. On this basis, it is shown that the widely used approximation considering the first order contribution from the variance of input variable can satisfactorily express the output uncertainty only when the input variance is very small or the input–output function is almost linear. Two applications of the analytic formula are performed to the power grid and economic systems where the sensitivities of both actual power output and Economic Order Quantity models are analyzed. The importance of each input variable in response to the model output is quantified by the analytic formula.

1. Introduction

Consider a deterministic model $y = f(\mathbf{X})$ with $\mathbf{X}$ indicating a multivariate vector. When y is calculated from $\mathbf{X}$ through a specified function, uncertainties in the input variables will propagate through the calculation to the output y [1,2]. This process is called variance propagation (or uncertainty propagation). Variance propagation, which is regarded as the basis of

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sensitivity analysis for complex models, mainly considers the determination of output's variance via uncertainties in input variables [3,4].

Many methods have been proposed for variance propagation, such as simulation-based methods [5,6], most probable point-based methods [7,8], functional expansion-based methods [9], numerical integration-based methods [10–13]. Simulation-based methods, also called sampling-based methods, are regarded as both effective and widely used, especially for those models without specific correspondence between y and $\mathbf{X}$ [14–16]. These methods, however, are computationally expensive, especially in the presence of a high number of input variables. For a general model with specific functional relationship between y and $\mathbf{X}$, the process will be much easier and numerically cheaper for determining the output's variance if an analytic formula associated with variance propagation can be provided. More information associated with other methods for variance propagation can be found in the reviewed papers [17–19].

A simple analytic formula has been appeared since 1953 which approximately computes the variance of the product of two independent random variables [20]. In 1966, this approximation was extended by engineers and experimentalists to more general multivariate cases [21]. This formula, also called Taylor series approximation, restricted to first-order terms [22], has gained a wide applications thanks to its simplicity and convenience [23]. However, it can satisfactorily estimate the output's variance only when the functional relationship between output and input variables is almost linear or the variance of each input variable is very small [17]. For most models, however, y highly nonlinearly depends on $\mathbf{X}$ having large uncertainties. This suggests the necessity of proposing an analytic formula to exactly calculate the output's variance and then to study its sensitivities in response to different input variables.

In the present paper, an analytic formula for variance propagation is proposed based on Taylor series expansion (univariate case is firstly considered) allowing to exactly determine the output uncertainty as well as the contributions of different orders from input uncertainty according to the output's derivatives and input's central moments. This formula is then extended to the general multivariate case followed by the applications in sensitivity and reliability analyses of two complex systems.

The paper is organized as follows: Section 2 shows the derivation of the analytic formula for variance propagation and its implementation in different univariate nonlinear functions. The analytic formula is extended to the general multivariate situation in Section 3. Section 4 presents applications of the extended formula in the analyses of two complex systems. Section 5 summarizes the results.

2. Analytic expression for variance propagation

Beginning with the univariate function, namely $y = f(x)$, its Taylor series expansion about a point $x = \mu$ is provided by

$$y = f(\mu) + \sum_{i=1}^{n} \frac{1}{i!} \left(\frac{\mathrm{d}^i f}{\mathrm{d}x^i} \right) (x-\mu)^i, \tag{1}$$

in which μ indicates the mathematical expectation of x. The above equation holds for a general function connecting y and x by making n go to infinity. Taking the average of both sides of Eq. (1) yields

$$E(y) = f(\mu) + \sum_{i=1}^{n} \frac{1}{i!} \left(\frac{\mathrm{d}^i f}{\mathrm{d}x^i} \right) \mu_i, \tag{2}$$

where μ_i is the ith central moment of variable x with definition given by

$$\mu_i = \int (x-\mu)^i P(x)\,\mathrm{d}x. \tag{3}$$

$P(x)$ labels the probability density function of x. The variance of y, say $V(y)$, then can be stated as

$$V(y) = \sum_{i,j=0}^{n} \frac{1}{i! \times j!} \left(\frac{\mathrm{d}^i f}{\mathrm{d}x^i} \times \frac{\mathrm{d}^j f}{\mathrm{d}x^j} \right) (\mu_{i+j} - \mu_i \mu_j). \tag{4}$$

This formula cannot only identify the contributions of different orders of the uncertainty in x with considering different values of n, but also exactly determine the output's variance by making n large enough. While $n = 1$, Eq. (4) only retains the first order contribution from the variance of x, indicated as $V(x)$, yielding

$$V(y) \approx \left(\frac{\mathrm{d}f}{\mathrm{d}x} \right)^2 V(x), \tag{5}$$

with $\mu_1 = 0$ and $\mu_2 = V(x)$ used. Eq. (5), called the general Taylor series expansion truncated to the first order, is most widely used to approximately calculate the uncertainty in y based on the mean and variance of x. This approximation, however, is satisfactory in the frequent case of highly nonlinear functions only when the variance of input is very small [17].