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Convex risk measures based on generalized lower deviation and their applications

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ABSTRACT

Considering the implementability and the properties that a reasonable and realistic risk measure should satisfy, we propose a new class of risk measures based on generalized lower deviation with respect to a chosen benchmark. Besides convexity and monotonicity, our new risk measure can reflect the investor's degree of risk aversion as well as the fat-tail phenomenon of the loss distribution with the help of different benchmarks and weighted functions. Based on the new risk measure, we establish a realistic portfolio selection model taking market frictions into account. To examine the influence of the benchmarks and weighted functions on the optimal portfolio and its performance, we carry out a series of empirical studies in Chinese stock markets. Our in-sample and out-of-sample results show that the new risk measure and the corresponding portfolio selection model can reflect the investor's risk averse attitude and the impact of different trading constraints. Most importantly, with the new risk measure we can obtain an optimal portfolio which is more robust and superior to the optimal portfolios obtained with the traditional expected shortfall risk measures.

1. Introduction

Investors determine on optimal portfolio for the purpose of increasing investment benefits and reducing the investment risk. Each investor has his own preference for risk and return, so he/she chooses a specific risk measurement model to measure the investment risk. The classical mean-variance (MV) model proposed by Markowitz (1952) has limited generality since it can only find optimal decisions if utility functions are quadratic or if investment returns are jointly elliptically distributed. However, quadratic utility implies the increasing absolute risk aversion over the whole domain. In addition, the real financial return data, with significant skewness and fat-tailness, do not follow the elliptical distribution. Therefore, MV model is not suitable for the actual financial industry.

The MV model and earlier portfolio selection models treat returns above and below the expected return equally. Nevertheless, many researches, such as Kahneman, Knetsch, and Thaler (1990), show that the attitude of investors towards profit and loss is asymmetric. Thus, any reasonable measure of investment risk should deal with downside risk and upside risk differently. In reality, most of investors are rational and risk averse (Sortino & Satchell, 2001). They mainly concern the downside risk. The classical downside risk measure is the lower partial moment (LPM) measure proposed by Bawa (1975). In recent years, with the introduction of value-at-risk (VaR), there has been a great momentum in research on quantile-based risk measures. Unfortunately, due to the lack of sub-additivity, VaR usually offers overly prudent market risk assessments (Pérignon & Smith, 2010a, 2010b); when calculated using scenarios, VaR is often nonconvex, non-smooth as a function of investment positions and is, therefore, difficult to optimize. Consequently, VaR cannot be considered as a reasonable risk measure, and is said to be seductive but dangerous (Bender, 1995) .

Aimed at providing a comprehensive theory of consistent measure of risk, Artzner, Delbaen, Eber, and et al. (1999) introduce the notion of coherent risk measure, which is further extended by Delbaen (2002) and Artzner, Delbaen, Eber, and et al. (2007) to more general setups. A measure is called a coherent risk measure if and only if it satisfies the following four axioms: monotonicity, positive homogeneity, subadditivity and translation invariance. It is easy to demonstrate that VaR does not provide coherency. Typical coherent risk measures include the expected shortfall (ES) defined in Acerbi and Tasche (2002); the conditional value-at-risk (CVaR) developed in Rockafellar and Uryasev (2002). The most general theoretical result about this kind of measures is the spectral measure defined in Acerbi (2002).

Nevertheless, the coherent risk measure is not widely applied in the financial management because of the limitations of the four axioms. For example, positive homogeneity means the risk value

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varies linearly with respect to the change of portfolio weights. But the limited liquidity in real markets makes it difficult to ensure positive homogeneity, because the risk value grows non-linearly with respect to portfolio weights (Ding, Shawky, & Tian, 2009; Chung & Hrazdil, 2010). Similarly, positive homogeneity is not realistic since it corresponds to linear utility functions, conflicting with the fact that the risk aversion of investors significantly grows when facing great losses. These facts imply the necessity to weaken the positive homogeneity (Bosch-Domènech & Silvestre, 2006). Thus, the framework of convex risk measures is introduced by Föllmer and Schied (2002). They replace the positive homogeneity and subadditivity with convexity. This class of risk measures has also attracted much attention (Lüthi & Doege, 2005) in recent years.

Another factor having a great influence on the consistent measure of risk is translation invariance. It deals with the change of risk measure when adding a constant to the random loss. Many researchers are skeptical about this. The problem is that the meaning of loss is ambiguous. The loss which Artzner et al. (1999) refer to is the negative cash flow of assets or portfolios. However, some financial specialists regard the loss as the return on assets which does not reach the expected target. So it is difficult to give a reasonable explanation to translation invariance. Besides, translation invariance means there is no disparity in how to distribute assets among the subsidiaries of a large financial corporation (Dhaene, Goovaerts, & Kaas, 2003). Apparently, the empirical assumption about asset allocation nowadays does not conform to the consistent measure of risk. Due to these reasons, deviation measures (Rockafellar, Uryasev, & Zabarankin, 2006), power CVaR (PCVaR) and some performance based measures (Farinelli, Ferreira, Rossello, & et al., 2008) are proposed. All these measures do not require the translation invariance.

Based on the above observations, Dhaene et al. (2003) argue that any reasonable measure of risk should satisfy monotonicity and convexity, which are widely recognized. Precisely because of the rationality and practicality of these two properties, we call it generalized convex risk measure and apply it to the actual financial decision-making problem.

As we can see, from the above demonstration, it is necessary to consider the downside risk. However, for risk measures such as VaR and CVaR, if we mainly care about big losses, the benefits will be reduced owing to overly conservative investment. Rockafellar et al. (2006) propose the generalized deviation to measure the uncertainty of the cash flow. By replacing the expected value in LPM with some given value, we can determine the corresponding lower deviation. The lower deviation can overcome the disadvantage that current tailed risk measures are not appropriate in bull market. Based on the lower deviation, Chen and Wang (2007) introduce a class of coherent risk measures based on p-norms and build an optimal portfolio selection model to show its superiority over CVaR. Later on, Chen and Wang (2008) construct a two-sided coherent risk measure based on the deviation of expectation.

Considering the practical application, what we mostly care is the practicality of risk measure for seeking an efficient portfolio. However, in most existing papers (Acerbi, 2002; Fischer, 2003; Rockafellar et al., 2006), the application of those risk measures to portfolio selection is not considered and the realistic portfolio selection problem under multiple market frictions is ignored.

Different from the above works, the aim of this application oriented paper is to build a new class of generalized convex lower deviation risk measures. By choosing a proper threshold value to identify the lower part of random cash flow, our new risk measure can overcome disadvantages of existing risk measures and reserve the widely-recognized monotonicity and convexity. The proposed measure can reflect the risk averse degree and control the loss distribution. Most importantly, the new risk measure is easy to apply to the optimal portfolio selection. We will construct the portfolio selection model which contains multiple market frictions and further show its advantages through empirical studies.

The paper is organized as follows: Section 2 introduces the new risk measure and examines its properties; Section 3 establishes a realistic portfolio selection model on the basis of the new risk measure by incorporating typical market frictions; Section 4 considers the application of the new risk measure in making optimal investment decisions by conducting empirical studies with real trading data from the Chinese stock markets; Section 5 concludes the paper.

2. Definition of new risk measure and its properties

In general, risk measurement can be regarded as the quantification of the characteristics of the future investment uncertainty. We consider the static investment framework, and it means that there exists a current time 0 and a future time D, and there is no trading between 0 and D. Thus risk can be expressed as a random cash flow X of some asset or portfolio which is defined on a probability space $(\Omega, \mathcal{F}, P)$ at time D. In this paper, we assume that X represents the uncertain return rate of some asset or portfolio in the future. So a risk measure is equivalent to a mapping ρ from some space of X to R. For convenience, we assume that X belongs to the space $L^p(\Omega, \mathcal{F}, P)$, here $1 \le p \le \infty$.

Based on the discussion in the previous section, we propose a new class of risk measures which considers the downside information of the loss distribution. With respect to a certain benchmark s, the random variable $(X-s)^-$ is the downside of X. In practical applications, we focus more on the downside than on the upside $(X-s)^+$. Thus we introduce a class of risk measures through taking a suitable nonlinear transformation to the downside of X. The new risk measure can not only reflect different investors' attitudes towards risk, but also satisfy convexity and monotonicity. These two properties are very important for a reasonable risk measure. The new risk measure is defined as follows:

Definition 1. [Generalized Lower Deviation Risk Measure, GLD_s for short]. For the random return X on $(\Omega, \mathcal{F}, P)$, the new risk measure, called the generalized lower deviation risk measure with respect to the given benchmark s, is defined as

$$GLD_s(X) = E\left[w((X-s)^-)\right]. \tag{1}$$

Here $w(x)$ is a non-negative, monotonically increasing and convex continuous function for $x \in [0, \infty)$.

Minimizing $GLD_s(X)$ is actually minimizing the nonlinearly weighted deviation with respect to the benchmark. This is the main difference between one-sided risk measures defined in Eq. (1) and two-sided coherent risk measures defined in Chen and Wang (2008). Compared with most coherent risk measures, $GLD_s(X)$ can be used to find more practical and robust optimal portfolios, which will be illustrated in Section 4. Now we interpret the flexibility and practicality of $GLD_s(X)$.

For professional investors, the loss is essentially the random cash flow lower than the chosen benchmark. So $GLD_s(\cdot)$ can be interpreted as measuring the nonlinearly weighted lower deviation. The proposed down-sided risk measure is appropriate for most investors with different investment psychologies. Depending on the choice of the weight function $w(x)$, $GLD_s(\cdot)$ nonlinearly treats different sizes of losses. Specifically, the higher the risk-averse degree the investor is, the more significant the convexity and nonlinearity of the function $w(x)$ should be; when $w(x) = x$, the risk measure turns into the ordinary lower deviation measure with respect to the chosen benchmark. Compared with $E(X)$ in the lower partial moment $\sigma^- = E[(X-E[X])^-]$ (Fischer, 2003; Rockafellar et al., 2006), the benchmark s is more