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Existence of optimal consumption strategies in markets with longevity risk[☆]

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ABSTRACT

Survival bonds are financial instruments with a payoff that depends on human mortality rates. In markets that contain such bonds, agents optimizing expected utility of consumption and terminal wealth can mitigate their longevity risk. To examine how this influences optimal portfolio strategies and consumption patterns, we define a model in which the death of the agent is represented by a single jump process with Cox–Ingersoll–Ross intensity. This implies that our stochastic mortality rate is guaranteed to be nonnegative, in contrast to many other models in the literature. We derive explicit conditions for existence of an optimal consumption and investment strategy in terms of model parameters by analysing certain inhomogeneous Riccati equations. We find that constraints must be imposed on the market price of longevity risk to have a well-posed problem and we derive the optimal strategies when such constraints are satisfied.

1. Introduction

This paper investigates the optimal consumption and asset allocation of an investor in a market which contains financial assets and contracts that are sensitive to longevity risk. These contracts, with a payoff that depends on the realized mortality rate in a large population, can be thought of as insurance products that can help to mitigate the effects of changes in survival probabilities during the lifetime of the agent. There is at the moment no liquid market for such products, but by introducing them in an asset allocation optimization problem we formulate a consistent and arbitrage-free way of analysing the influence of the market price of longevity risk on investment behaviour. The precise value of such a market price of risk may be difficult to estimate in practice, but it is not realistic to put it at zero, especially for the analysis of retirement provisions.

The classical formulation of the optimal consumption and investment problems that we wish to consider here goes back to Merton (1969, 1971). In that setup, markets are complete and there is a fixed time period for investment and consumption. These results have been extended in many directions, by changing preferences as in Musiela and Zariphopoulou (2010) and Kraft et al. (2011), or by specifying different assumptions on asset price dynamics. Some authors, including Kraft (2005) and Chacko and Viceira (2005), have considered stochastic volatility processes for equity prices and solved the optimization problem for that case. Others have introduced more realistic fixed income markets by introducing stochastic interest rates. Many papers use Gaussian models for the short rate but Deelstra et al. (2000) and Kraft (2005, 2009) derive the optimal strategy for an agent maximizing power utility from terminal wealth by investing in a market with a short-rate following a Cox–Ingersoll–Ross (CIR) process, which will thus remain positive at all times.

If the investment horizon is uncertain, the optimal strategies need to be adjusted. We can distinguish between problems where the end of the investment period is chosen by the agent himself, such as the optimal stopping problem for flexible retirement in Dybvig and Liu (2010), and problems where the end of the investment period cannot be chosen by the agent. Models in which the death of the agent is included in the model form an obvious example of the latter category. In that case agents face a trade-off between obtaining an amount of utility now with certainty versus an amount of utility in the future which is uncertain due to both financial risk and mortality risk. Yaari (1965) seems to be the first to consider such consumption and investment problems in which the lifetime of the agent is stochastic. Under the assumption that the probability of death at a given age is

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known and constant in time, he solves the optimal investment problem with uncertain lifetime in continuous time using dynamic programming methods. The solution shows that one can interpret the deterministic mortality rate in terms of adjusted discount rates. Hakansson (1969) obtained similar results in a discrete time. Pliska and Ye (2007), building on previous work by Richard (1975), introduced life insurance as an extra asset for investment. They derive closed form solutions for the investment in stocks, bonds and life insurance products under the assumption that mortality rates are time-varying but deterministic. Huang and Milevsky (2008) extended these results to hyperbolic absolute risk aversion (HARA) preferences and include a stochastic income process. In Charupat and Milevsky (2002) and Milevsky and Young (2007) annuities instead of life insurance contracts are added to the asset mix.

All these models use time-varying but deterministic mortality rates. One of the first authors to study optimal consumption strategies when mortality rates are stochastic is Menoncin (2008). He solves the Hamilton–Jacobi–Bellman equation associated with the optimization problem of an investor who is exposed to a stochastic mortality rate while allowing for a general specification of the asset price dynamics. A longevity bond is available in the economy to mitigate the effects of this risk. Blanchet-Scalliet et al. (2008) allow the conditional distribution function of an agent's remaining time-horizon to be stochastic and correlated to asset returns.[1] To get closed form expressions, the rate which determines the random time at which the investor must liquidate the asset portfolio is allowed to become negative with positive probability. Huang et al. (2012) compare the optimal consumption strategy for an agent with a deterministic versus a stochastic mortality rate in a market where only the money market account is available for investment. They show that, compared to the deterministic case, agents increase or decrease their initial consumption in the stochastic case, depending on whether their constant risk aversion coefficient makes them more or less risk averse than an investor with a logarithmic utility function. In more recent work Shen and Wei (2014) solve an optimal consumption and asset allocation problem in a general diffusion framework under the premise that an exponential integrability condition is satisfied. Guambe and Kufakunesu (2015) generalize these results to Lévy processes.

As an extension of these results we provide conditions that guarantee existence of an optimal consumption strategy in a market where rates follow CIR dynamics. We will thus model mortality by a stochastic process, which will almost surely stay positive at all times. Affine mortality rates lead to analytically tractable survival probabilities which remain smaller than one, as was noted earlier in work by Biffis (2005) and Schrager (2006); see also Dahl (2004). We include both financial assets and survival bonds in the asset mix and allow a stochastic market price of mortality risk. The optimal consumption and investment strategies are derived in semi-closed form and we show that the hedging demand in our model is bounded. The impact of stochastic mortality on initial consumption is characterized in terms of the risk aversion coefficient.

The existence of a solution to the problem studied in the present paper depends on the asymptotic behaviour of the utility function for large values of wealth and the existence of moments of the state-price density. It was noted in Korn and Kraft (2004) and Kraft (2009) that not all moments of the state-price density in a model based on a CIR short rate may be well-defined. Therefore the present paper takes a different approach than the one in Shen and Sherris (2014) and Menoncin and Regis (2015), since we want to characterize explicitly under which conditions optimal strategies exist. By assuming that the market prices of risk are proportional to the square root of the short rate and the mortality rate (with a time-varying proportionality coefficient) we can derive such conditions by analysing the existence and uniqueness of bounded solutions for certain inhomogeneous Riccati equations.

The remainder of this paper is organized as follows. Section 2 introduces the model for the economy. In Section 3 the investment problem is formulated and the main result of the paper is presented. Section 4 provides a characterization of the Laplace transform of a Cox–Ingersoll–Ross process which is needed to prove the main result. The proof of the main result is given in Section 5. The economic implications of the model are discussed in Section 6. Section 7 concludes.

2. The model for the economy

In this section we will construct a complete financial market in which asset returns, interest rates and mortality rates are uncertain. Let $(\Omega, \mathcal{G}, P)$ be a probability space on which an exponentially distributed random variable Θ and a three-dimensional standard Brownian motion $W(t) = (W_1(t), W_2(t), W_3(t))'$ are defined. We study the optimal investment and consumption strategy of an investor during the timespan $[0, T]$, for some $0 < T < \infty$, and we allow for the possibility that the investor does not survive until T. Let $\{\mathcal{F}(t),\ t \in [0, T]\}$ be the P-augmentation of the filtration generated by the process W. The exponential random variable Θ is taken to be independent of $\mathcal{F}(T)$.

To model the time of death of the investor we introduce the nonnegative, $\mathcal{G}$-measurable random time

$$\tau = \inf\left\{t \geq 0 : \int_0^t \lambda(u)\, du \geq \Theta\right\}, \tag{2.1}$$

in which the so-called mortality rate $\lambda(t)$ follows a Cox–Ingersoll–Ross process, that is,

$$\lambda(t) = \lambda_0 + \int_0^t (\mu_2(u) - \kappa_2\, \lambda(u))\, du + \int_0^t \xi_2 \sqrt{\lambda(u)} dW_2(u). \tag{2.2}$$

The constants λ_0, κ_2, ξ_2 and μ_2 are assumed to be bounded and strictly positive, and μ_2 is a deterministic, continuously differentiable function on $[0, T]$. The extended Feller condition[2]

$$2\,\mu_2(t) \geq \xi_2^2, \quad \text{for all } t \in [0, T], \tag{2.3}$$

ensures that the mortality rate is strictly positive almost surely. Since $\int_0^t \lambda(u)\, du$ is $\mathcal{F}(t)$-measurable and Θ is independent of $\mathcal{F}(T)$, the survival probability of the agent satisfies

$$\begin{aligned}\overline{F}(t) &:= P[\tau > t \mid \mathcal{F}(t)] = P\left[\int_0^t \lambda(u)\, du < \Theta \,\middle|\, \mathcal{F}(t)\right] \\ &= \exp\left(-\int_0^t \lambda(u)\, du\right).\end{aligned} \tag{2.4}$$

To finance his/her consumption and long-term wealth objectives, the agent can invest in a number of assets: a stock, a zero-coupon bond, a longevity bond and the money-market account.

The value β of the money market account, based on the continuously compounded stochastic short rate r, satisfies

$$\beta(t) = 1 + \int_0^t \beta(u) r(u) du.$$

[1] See Maurer (2011) for another example of markets where mortality rates are correlated with asset returns.

[2] See Theorem 4.2 in Maghsoodi (1996).