European Journal of Operational Research 256 (2017) 292–307

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Interfaces with Other Disciplines

Bayesian estimation of the global minimum variance portfolio[☆]

Taras Bodnar[a], Stepan Mazur[b], Yarema Okhrin[c,*]

[a] Department of Mathematics, Stockholm University, Roslagsvägen 101, Stockholm SE-10691, Sweden
[b] Department of Statistics, Lund University, Tycho Brahes väg 1, Lund SE-22007, Sweden
[c] Department of Statistics, University of Augsburg, Augsburg D-86159, Germany

ARTICLE INFO

Article history:
Received 20 April 2015
Accepted 25 May 2016
Available online 21 June 2016

Keywords:
Global minimum variance portfolio
Posterior distribution
Credible interval
Wishart distribution

ABSTRACT

In this paper we consider the estimation of the weights of optimal portfolios from the Bayesian point of view under the assumption that the conditional distributions of the logarithmic returns are normal. Using the standard priors for the mean vector and the covariance matrix, we derive the posterior distributions for the weights of the global minimum variance portfolio. Moreover, we reparameterize the model to allow informative and non-informative priors directly for the weights of the global minimum variance portfolio. The posterior distributions of the portfolio weights are derived in explicit form for almost all models. The models are compared by using the coverage probabilities of credible intervals. In an empirical study we analyze the posterior densities of the weights of an international portfolio.

1. Introduction

Starting with the seminal paper of Markowitz (1952) the classical mean-variance portfolio theory has drawn much attention in academic literature. Generally speaking, the theory allows us to determine the optimal port/folio weights which guarantee the lowest risk for a given expected portfolio return. Under Gaussian asset returns, the problem is equivalent to minimizing the expected quadratic utility of the future wealth (c.f., Bodnar, Parolya, & Schmid, 2013). In practice, however, the model frequently led to investment opportunities with modest ex-post profits and high risk. To clarify this and to develop improved trading strategies several issues were addressed, which can be roughly separated in two partly overlapping branches in literature. The first strand of research analyses the estimation risk in portfolio weights, which arises if we replace the unknown parameters of the distribution of asset returns with their sample counterparts. The results on the finite sample distributions can be used in different ways. First, we can develop a test to check if the weights of a particular asset significantly deviate from prespecified values, e.g. test for efficiency (see Ang & Bekaert, 2002; Bodnar & Schmid, 2008; Britten-Jones, 1999; Jobson & Korkie, 1989; Stambaugh, 1997). Second, we can test the significance of the investment in a given asset, e.g. significance of international diversification (see French & Poterba, 1991). Third, we may assess the sensitivity of portfolio weights to changes in the parameters of the asset returns as in Best and Grauer (1991); Bodnar (2009); Chopra and Ziemba (1993), and many others.

The main contribution of Markowitz from the financial perspective is the recognition of the importance of diversification. From a statistical point of view, the portfolio theory stresses the importance of the variance as a measure of risk and particularly the importance of the structure of the covariance matrix for diversification purposes. Markowitz's approach allows us to determine the minimum variance set of portfolios and the sets of efficient portfolios. While the minimum variance set consists of those portfolios which possess the minimum variance for a chosen level of the expected return, the efficient set contains the portfolios with the highest level of the expected return for each level of risk. As a result, the choice of an optimal portfolio depends on the investor's attitude towards risk, i.e. on his/her level of risk aversion. Markowitz (2014) showed both theoretically and empirically that the mean-variance method and the expected utility approach lead to similar optimal portfolios, whereas (Liesiö & Salo, 2012) developed a portfolio selection framework which uses the set inclusion to capture incomplete information about scenario probabilities and utility functions. This approach identifies all of the non-dominated project portfolios in view of this information as well as it offers the decision support for the rejection and selection of projects.

☆ The authors appreciate the financial support of the German Science Foundation (DFG), projects BO3521/2 and OK103/1, "Wishart Processes in Statistics and Econometrics: Theory and Applications". The first author is partly supported by the German Science Foundation (DFG) via the Research Unit 1735 "Structural Inference in Statistics: Adaptation and Efficiency". He also appreciates the financial support of the German Science Foundation (DFG) via the projects BO 3521/3-1 and SCHM 859/13-1 "Bayesian Estimation of the Multi-Period Optimal Portfolio Weights and Risk Measures".

* Corresponding author. Tel.: +49 821 5984152; fax: +49 821 5984227.
E-mail address: yarema.okhrin@wiwi.uni-augsburg.de (Y. Okhrin).

http://dx.doi.org/10.1016/j.ejor.2016.05.044
0377-2217/

Levy and Levy (2014) analyzed the impact of estimation error in portfolio optimization, while (Bodnar, Parolya, & Schmid, 2015a; 2015c) present analytical solutions to multi-period portfolio choice problems based on the quadratic and exponential utility functions.

The global minimum variance (GMV) portfolio is a specific optimal portfolio which possesses the smallest variance among all portfolios on the efficient frontier. This portfolio corresponds to the fully-risk averse investor who aims to minimize the variance without taking the expected return into consideration. The importance of the GMV portfolio in financial applications was well motivated by Merton (1980) who pointed out that the estimates of the variances and the covariances of the asset returns are much more accurate than the estimates of the means. Later, Best and Grauer (1991) showed that the sample efficient portfolio is extremely sensitive to changes in the asset means, whereas (Chopra & Ziemba, 1993) concluded for a real data set that errors in means are over ten times as damaging as errors in variances and over twenty times as errors in covariances. For this reason many authors assume equal means for the portfolio asset returns or, in other terms, the GMV portfolio. This is one reason why this is extensively discussed in literature (Chan et al. 1999). Moreover, the GMV portfolio has the lowest variance of any feasible portfolio. More evidence regarding the practical application of the GMV portfolio can be found in Haugen (1999).

In contrary to the above approaches, the second strand of research opts for the Bayesian framework. The Bayesian setting resembles the decision making of market participants and the human way of information utilization. Similarly, investors use the past experiences and memory (historical event, trends) for decisions at a given time point. These subjective beliefs flow into the decision making process in a Bayesian setup via specific priors. From this point of view the Bayesian framework is potentially more attractive in portfolio theory (see Avramov & Zhou, 2010). The first applications of Bayesian statistics in portfolio analysis were completely based on uninformative or data-based priors, see Winkler (1973); Winkler and Barry (1975). Bawa, Brown, and Klein (1979) provided an excellent review on early examples of Bayesian studies on portfolio choice. These contributions stimulated a steady growth of interest in Bayesian tools for asset allocation problems. Barberis (2000); Jorion (1986); Kandel and Stambaugh (1996); Pastor (2000) used the Bayesian framework to analyze the impact of the underlying asset pricing or predictive model for asset returns on the optimal portfolio choice. Bodnar, Parolya, and Schmid (2015b); Golosnoy and Okhrin (2007); 2008); Kan and Zhou (2007); Wang (2005) concentrated on shrinkage estimation, which allows to shift the portfolio weights to prespecified values, reflecting the prior beliefs of investors. Brandt (2010) gives a state of the art review of modern portfolio selection techniques, paying particular attention to Bayesian approaches.

In the majority of the mentioned papers, the authors defined specific priors for the model parameters and the subsequent evaluation of posterior distributions or asset allocation decisions was performed numerically. The reason is that the involved integral expressions are too complex for analytic derivation. In this paper we derive explicit formulas for the posterior distributions of the global minimum-variance portfolio weights for several non-informative and informative priors on the parameters of asset returns. Furthermore, using a specific reparameterization we obtain non-informative and informative priors for the portfolio weights directly. This appears to be more consistent with the decision processes of investors. The corresponding posterior distributions are presented too. The established results are evaluated within a simulation study, which assesses the coverage probabilities of credible intervals, and within an empirical study, where we concentrate on the posterior distributions of the weights of an internationally diversified portfolio.

The rest of the paper is structured as follows. Bayesian estimation of the GMV portfolio using preliminary results is presented in Section 2. The posterior distributions for the GMV portfolio are derived and summarized in Theorem 1. In Section 3 we propose informative and non-informative prior distributions for the weights of the GMV portfolio and the corresponding posterior distributions (Theorems 2 and 3). In Section 4 the credible intervals and credible sets for the previous posterior distributions are obtained. The results of numerical and empirical studies are given in Section 5, while Section 6 summarizes the paper. The appendix (Section 7) contains the proof of Theorem 1 and additional technical results.

2. Bayesian portfolio selection

We consider a portfolio of k assets. Let $\mathbf{X}_i = (X_{1i}, \dots, X_{ki})^T$ be the k-dimensional random vector of log-returns at time $i = 1, \dots, n$. For small values of returns, the simple and the log-returns behave similarly. Let $\mathbf{w} = (w_1, \dots, w_k)^T$ be the vector of portfolio weights, where w_j denotes the weight of the jth asset, and let $\mathbf{1}$ be the vector of ones. Assuming that dynamics of the terminal wealth is governed by the standard Brownian motion[1], we obtain the log-normal distribution as the distribution of the wealth. This leads to Gaussian portfolio log-returns, which are equal to the sum of the log-returns of the underlying assets, i.e. $\mathbf{X_w} = \sum_{i=1}^k w_i \mathbf{X}_i$ (see Dhaene, Vanduffel, Goovaerts, Kaas, & Vyncke, 2005). Note that below we assume a conditional normal distribution of the asset returns given the mean vector and the covariance matrix, which is a much weaker assumption than the unconditional Gaussian distribution. Let the mean vector of the asset returns be denoted by $\boldsymbol{\mu}$ and the covariance matrix by a positive definite matrix $\boldsymbol{\Sigma}$. The GMV portfolio is the unique solution of the optimization problem

$$\mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w} \rightarrow min \quad \text{s.t.} \quad \mathbf{w}^T \mathbf{1} = 1. \tag{1}$$

In general we allow for short sales and therefore for negative weights. The solution of (1) is given by

$$\mathbf{w}_{GMV} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^T \boldsymbol{\Sigma}^{-1}\mathbf{1}}. \tag{2}$$

Since $\boldsymbol{\Sigma}$ is an unknown parameter, the formula in (2) is infeasible for practical purposes. Given a sample of size n of historical vectors of returns $\mathbf{x}_1, \dots, \mathbf{x}_n$, we can compute the sample covariance matrix

$$\mathbf{S} = \frac{1}{n-1} \sum_{i=1}^{n} (\mathbf{x}_i - \overline{\mathbf{x}})(\mathbf{x}_i - \overline{\mathbf{x}})^T,$$

where $\overline{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i$. The sample estimator of the GMV portfolio weights is constructed by replacing $\boldsymbol{\Sigma}$ with $\mathbf{S}$ in (2) and it is given by

$$\widehat{\mathbf{w}}_{GMV} = \frac{\mathbf{S}^{-1}\mathbf{1}}{\mathbf{1}^T \mathbf{S}^{-1}\mathbf{1}}. \tag{3}$$

In this paper we take a more general setup by considering arbitrary linear combinations of the GMV portfolio weights. Let $\mathbf{L}$ be an arbitrary $p \times k$ matrix of constants, $p < k$, and define

$$\boldsymbol{\theta} = \mathbf{L}\mathbf{w}_{GMV} = \frac{\mathbf{L}\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^T \boldsymbol{\Sigma}^{-1}\mathbf{1}}. \tag{4}$$

[1] In fact when the length of the subsequent time intervals becomes smaller we obtain in the limit that $(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n)$ effectively becomes increments of a multi-dimensional Brownian motion and by keeping the portfolio weights constant during the investment horizon (by continuously rebalancing) we obtain that the terminal wealth is lognormally distributed. In this regard, note that the assumption of constant portfolio weights appears as a trading constraint. There is also a rich literature on optimal mean-variance portfolios when there are no such trading constraints (in which case the optimal terminal wealth is no longer lognormally distributed); see, Bernard and Vanduffel (2014); Goetzmann, Ingersoll, Spiegel, and Welch (2007)