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Portfolio selection based on a benchmark process with dynamic value-at-risk constraints

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A B S T R A C T

Portfolio selection is an essential issue in finance. It studies how to allocate one's wealth in a basket of securities to maximize the return and minimize the risk. And dynamic portfolio selection based on a benchmark process is one of the most important types. Different from the existing literature, we impose a dynamic risk control on it. As a matter of fact, performing an optimal portfolio strategy in the light of a dynamic portfolio formulation does not eliminate the possibility of an investor going to bankruptcy or even more serious situations in a volatile financial market before the terminal time, so it is reasonable and necessary to impose a dynamic risk control on the instantaneous wealth throughout the investment horizon to ensure that the investment behavior can proceed and we intend to address this interesting issue in this paper. More specifically, we investigate the dynamic portfolio selection problem based on a benchmark process coupled with a dynamic value-at-risk constraint. By stochastic dynamic programming techniques, we derive the corresponding Hamilton–Jacobi–Bellman equation. Moreover, the optimal portfolio strategies are obtained by Lagrange multiplier method. To verify the model, two numerical examples are illustrated. The results show the difference of optimal portfolio strategies with and without the dynamic VaR constraint: the composition of the risky assets is constant but the investment proportion is reduced as the VaR constraint becomes binding. This research can provide a good decision-making reference for risk-averse investors.

1. Introduction

Markowitz's mean–variance model marked a new epoch in portfolio theory [1]. In his seminal work, the buy-and-hold portfolio strategy is considered and variance is used as a risk measure. However, in most cases, investment is a long-term dynamic process. Investors have to adjust their portfolio strategies continuously during investment horizon according to the actual financial market conditions. Merton [2] and Samuelson [3] extended Markowitz's model to a continuous time one by stochastic control theory respectively, in which they maximized the expected utility function. Afterwards, expected utility maximization model became a standard framework of dynamic portfolio selection problem. And many scholars studied the problem further by imposing different constraints, such as [4–7]. However, not all investors believe that there exist suitable utility functions to reflect their investment preferences. In practice, many investors intend to seek a balance between the expected terminal wealth and some risk measure or to track a benchmark process as closely as possible. In 2000, Markowitz's static model was extended by Li et al. [8] and Zhou et al. [9] to the dynamic case, respectively, in a discrete-time

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and continuous-time frameworks. Since then, dynamic mean–variance portfolio selection has become a hot research area, see [10–13]. The benchmark process is an important index in most real world situations and portfolio selection based on it is arousing more and more interests and attentions, see [14–18]. In this case, the investor's objective is to approach or track an expected wealth process or some index as closely as possible.

As many literature pointed out, variance is not a good risk measure since it takes deviation both up and down from the mean without discrimination as the risk. As its modification, many risk measures, such as semi-variance, mean absolute deviation [19], value-at-risk (VaR) [20], conditional value-at-risk [21], etc., have been proposed. Among all risk measures, VaR is appealing by its simplicity and clarity. It has even been recommended as a standard on banking supervision by the Basel Committee. By definition, VaR is the maximum loss of a portfolio at a certain confidence level over a given time horizon, which is a percentile in fact. Though VaR is criticized by its loss of subadditivity, it is still widely adopted in practice.

In this paper, we focus on a continuous time portfolio selection problem based on a benchmark process coupled with a dynamic relative VaR constraint. This problem is of great realistic significance. In fact, in order to prevent investors from extremely dangerous positions, which may result in bankruptcy or even more serious situations, it is reasonable and evidently needed to employ asymmetric risk measures (e.g., VaR) to control the exposure to market risks throughout the investment horizon. And to our knowledge, this problem has not been studied yet in the existing literature.

The rest of this paper is organized as follows. In Section 2, we describe the market setting and problem formulation. Meanwhile, the explicit expression of VaR is derived. In Section 3, we solve the model by dynamic programming techniques. Moreover, the closed-form solution of the optimal portfolio strategy and the explicit expression of the optimal value function are presented. In Section 4, we illustrate two numerical examples to show how the addition of the VaR constraint affects the optimal strategy. At last, we summarize the paper.

2. Market setting and problem formulation

Consider a financial market with one risk-free asset and n risky assets. An investor enters the market with initial wealth x_0 and invests in these $n+1$ assets continuously within time horizon $[0, T]$, where T is a finite positive number, representing the terminal time of the investment. Assume all the randomness is modeled by $(\Omega, F, P, \{F_t\}_{t\geq 0})$, a filtered complete probability space, on which an F_t-adapted n-dimensional Brownian motion $B(t) = (B_1(t), , \dots, B_n(t))'$ is defined, where $B_i(t)$ and $B_j(t)$ are mutually independent for any $i \neq j$. The price of the risk-free asset $P_0(t)$ satisfies the following ordinary differential equation

$$dP_0(t) = P_0(t)rdt, \qquad P_0(0) = p_0,$$

where $r > 0$ is the rate of interest and is assumed to be constant. The other n risky-assets' prices $P(t) = (P_1(t), \dots, P_n(t))'$ follow the following stochastic differential equations

$$dP_i(t) = P_i(t)\left[\mu_i dt + \sum_{j=1}^{n} \sigma_{ij} dB_j(t)\right], \qquad P_i(0) = p_i, \quad i = 1, \dots, n,$$

where $\mu = (\mu_1, \dots, \mu_n)'$ is the appreciation rate vector, $\sigma = (\sigma_{ij})_{n\times n}$ is the volatility matrix. Moreover, here $\mu > 0$ and σ are assumed to be constant and σ is invertible. Let $L_F^2(0, T; R^n)$ be the set of all R^n- valued, F_t-adapted and square integrable stochastic processes. Assume $\pi(t) = [\pi_1(t), \dots, \pi_n(t)]'$ be the portfolio process with $\pi(\cdot) \in L_F^2(0, T;\ R^n)$, where $\pi_i(t)$ is the fraction of the wealth invested in the ith risky asset at time $t \in [0, T]$. Then the investment proportion of the risk-free asset is $1 - \sum_{i=1}^{n} \pi_i(t)$. Thus, the wealth process $\{X(t), t \in [0, T]\}$ satisfies

$$dX(t) = X(t)\{[r + \pi'(t)(\mu - r1_n)]dt + \pi'(t)\sigma dB(t)\}, \qquad X(0) = x_0, \tag{2.1}$$

where $1_n = (1, \dots, 1)'$ is a n-dimensional vector with all elements of 1. Rewriting (2.1) into an integral form by Ito formula, we have

$$X(t) = x_0 \exp\left\{\int_0^t \left[r + \pi'(s)(\mu - r1_n) - \frac{1}{2}\left\|\pi'(s)\sigma\right\|^2\right] ds + \int_0^t \pi'(s)\sigma dB(s)\right\}. \tag{2.2}$$

Let $y(t), t \in [0, T]$ be a benchmark process, representing the expected wealth process, which is a given function. Take the difference of the wealth process and the benchmark process as the residual process. In this paper, we assume the investor's objective is to minimize the cumulative variance of the residual process, that is

$$\min_{\pi(t)\in \mathrm{U}([0,T])} E\int_0^T (X(t) - y(t))^2 dt,$$

where $\mathrm{U}([0, T])$ is the set of all admissible portfolio strategies.

In what follows, we derive the analytical expression of VaR. The idea comes from [4]. By (2.2), we have

$$\begin{aligned} X(t+\tau) &= x_0 \exp\left\{\int_0^{t+\tau} \left[r + \pi'(s)(\mu - r1_n) - \frac{1}{2}\left\|\pi'(s)\sigma\right\|^2\right] ds + \int_0^{t+\tau} \pi'(s)\sigma dB(s)\right\} \\ &= X(t) \exp\left\{\int_t^{t+\tau} \left[r + \pi'(s)(\mu - r1_n) - \frac{1}{2}\left\|\pi'(s)\sigma\right\|^2\right] ds + \int_t^{t+\tau} \pi'(s)\sigma dB(s)\right\}, \end{aligned}$$