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Colorful linear programming, Nash equilibrium, and pivots

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ABSTRACT

The colorful Carathéodory theorem, proved by Bárány in 1982, states that given $d+1$ sets of points $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$ in $\mathbb{R}^d$, with each $\mathbf{S}_i$ containing $\mathbf{0}$ in its convex hull, there exists a set $T \subseteq \bigcup_{i=1}^{d+1} \mathbf{S}_i$ containing $\mathbf{0}$ in its convex hull and such that $|T \cap \mathbf{S}_i| \leq 1$ for all $i \in \{1, \ldots, d+1\}$. An intriguing question – still open – is whether such a set T, whose existence is ensured, can be found in polynomial time. In 1997, Bárány and Onn defined colorful linear programming as algorithmic questions related to the colorful Carathéodory theorem. The question we just mentioned comes under colorful linear programming.

The traditional applications of colorful linear programming lie in discrete geometry. In this paper, we study its relations with other areas, such as game theory, operations research, and combinatorics. Regarding game theory, we prove that computing a Nash equilibrium in a bimatrix game is a colorful linear programming problem. We also formulate an optimization problem for colorful linear programming and show that as for usual linear programming, deciding and optimizing are computationally equivalent. We discuss then a colorful version of Dantzig's diet problem. We also propose a variant of the Bárány algorithm, which is an algorithm computing a set T whose existence is ensured by the colorful Carathéodory theorem. Our algorithm makes a clear connection with the simplex algorithm and we discuss its computational efficiency. Related complexity and combinatorial results are also provided.

1. Introduction

1.1. Context

In 1982, Bárány proved a colorful generalization of the Carathéodory theorem, whose statement is the following.

Theorem 1 (*Colorful Carathéodory Theorem [2]*). *Given $d+1$ sets of points $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$ in $\mathbb{R}^d$ such that each $\mathbf{S}_i$ contains $\mathbf{0}$ in its convex hull, there exists a set T of the form $\{s_1, \ldots, s_{d+1}\}$, with $s_i \in \mathbf{S}_i$ for every $i \in \{1, \ldots, d+1\}$, that contains $\mathbf{0}$ in its convex hull.*

A natural question raised by this theorem is whether such a *colorful* set T can be computed in polynomial time. The case with $\mathbf{S}_1 = \cdots = \mathbf{S}_{d+1}$, corresponding to the usual Carathéodory theorem, is known to be solvable in polynomial time, via linear programming. However, the complexity of the colorful version remains an open question.

In 1997, Bárány and Onn defined algorithmic and complexity problems related to the colorful Carathéodory theorem [4], giving birth to colorful linear programming. In their paper, the complexity question raised by the colorful Carathéodory

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theorem is referred as an "outstanding problem on the borderline of tractable and intractable problems". In addition to providing a theoretical challenge, the colorful Carathéodory theorem has several applications in discrete geometry (e.g. Tverberg partition, "first selection lemma", see [22]). Any efficient algorithm computing such a colorful set T would benefit these applications.

A set of points is *positively dependent* if it is nonempty and contains $\mathbf{0}$ in its convex hull. Given a configuration of k sets of points $\mathbf{S}_1, \ldots, \mathbf{S}_k$ in $\mathbb{R}^d$, a set T is *colorful* if it is of the form $\{s_1, \ldots, s_k\}$ with $s_i \in \mathbf{S}_i$ for every $i \in \{1, \ldots, k\}$. One of the problems studied in this paper is the following.

Colorful Carathéodory
Input. A configuration of $d+1$ positively dependent sets of points $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$ in $\mathbb{Q}^d$.
Task. Find a positively dependent colorful set.

As we have already mentioned, the complexity status is still open. A more general problem has been recently proved to be PLS-complete by Mulzer and Stein [26]. The PLS class, where PLS stands for "Polynomial Local Search", contains the problems for which local optimality can be verified in polynomial time [15]. The original proof of the colorful Carathéodory theorem by Bárány naturally provides an algorithm solving Colorful Carathéodory. This algorithm, known as the Bárány algorithm, was analyzed and improved by Bárány and Onn [4]. It is a pivot algorithm roughly relying on computing the closest facet of a simplex to $\mathbf{0}$. Although not polynomial, this algorithm is quite efficient, as proved by Deza et al. through an extensive computational study [10]. In addition to Colorful Carathéodory, Bárány and Onn formulated the following problem, which is in a sense more general.

Colorful Linear Programming
Input. A configuration of k sets of points $\mathbf{S}_1, \ldots, \mathbf{S}_k$ in $\mathbb{Q}^d$.
Task. Decide whether there exists a positively dependent colorful set.

We emphasize that when we write "Colorful Linear Programming", with small capital letters, we refer to that problem, as in Bárány–Onn paper [4], but when we write "colorful linear programming", we mean the study of the family of all problems about finding or deciding the existence of a positively dependent colorful set.

Bárány and Onn showed that the case of Colorful Linear Programming with $k = d$ is NP-complete even if each $\mathbf{S}_i$ is of size 2, proving that the general case is NP-complete as well. It contrasts with Colorful Carathéodory. In this version, when each $\mathbf{S}_i$ is of size 2, we clearly have a polynomial special case: select one point in each $\mathbf{S}_i$, find the linear dependency, and change for the other point in $\mathbf{S}_i$ for those having a negative coefficient.

A slightly more general version of Colorful Carathéodory can be defined with conic hulls instead of convex hulls.

Colorful Carathéodory (conic version)
Input. A configuration of d sets of points $\mathbf{S}_1, \ldots, \mathbf{S}_d$ in $\mathbb{Q}^d$ and an additional point $\boldsymbol{p}$ in $\bigcap_{i=1}^d \operatorname{cone}(S_i) \cap \mathbb{Q}^d$.
Task. Find a colorful set T such that $\boldsymbol{p} \in \operatorname{cone}(T)$.

The colorful set T exists for sure because of a conic version of the colorful Carathéodory theorem, also proved by Bárány in the same paper [2]. By an easy geometric argument, this problem coincides with Colorful Carathéodory when $\operatorname{conv}(\bigcup_{i=1}^d \mathbf{S}_i)$ does not contain $\mathbf{0}$. Note that as usual for this kind of problems, there is a shift in the dimension when going from one version to the other.

We define in the same way a conic version of Colorful Linear Programming.

Colorful Linear Programming (conic version)
Input. A configuration of k sets of points $\mathbf{S}_1, \ldots, \mathbf{S}_k$ in $\mathbb{Q}^d$ and an additional point $\boldsymbol{p}$ in $\mathbb{Q}^d$.
Task. Decide whether there exists a colorful set T such that $\boldsymbol{p} \in \operatorname{cone}(T)$.

1.2. Main contributions

"Bimatrix **is a colorful linear programming problem**".

We prove that the problem Bimatrix, consisting in computing a Nash equilibrium in a bimatrix game, is polynomially reducible to Finding Another Colorful Simplex Problem, which is a colorful linear programming problem introduced by Meunier and Deza [25]. It shows that this latter problem is PPAD-complete. This was stated as an open question in the cited paper. On our way, we introduce a new method for proving that a linear complementarity problem belongs to the PPAD class, based on Sperner's lemma. This method seems to be interesting for its own sake, since it avoids the introduction of oriented primoids, as done usually. All these results are stated and proved in Section 2, where a definition of the PPAD class is also provided.

"**The simplex algorithm solves Colorful carathéodory**".

In Section 3, we show that the Bárány algorithm and its improvement by Bárány and Onn can be slightly modified in order to get what is more or less a "Phase I" simplex method. Instead of computing a closest facet at each pivot step, we select a new point by a classical reduced cost consideration. It simplifies the iteration, and improves the complexity of this latter. A complexity analysis of the overall algorithm is provided, as well as numerical experiments that show the effectiveness of the approach.

"**Optimization and decision are equivalent for colorful linear programming**".

For the usual linear programming, it is known that the problem of deciding the existence of a solution to a linear program and the problem of optimizing a linear program are polynomially equivalent, see for instance Theorem 10.4 of Schrijver [30].