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Solving the interval linear programming problem: A new algorithm for a general case

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ABSTRACT

Based on the binding constraint indices of the optimal solution to the linear programming (LP) model, a feasible system of linear equations can be formed. Because an interval linear programming (ILP) model is the union of numerous LP models, an interval linear equations system (ILES) can be formed, which is the union of these conventional systems. Hence, a new algorithm is introduced in which an arbitrary characteristic model of the ILP model is chosen and solved. The set of indices of its binding constraints is then obtained. This set is used to form and solve an ILES using the enclosure method. If all the components of the interval solutions to this system are strictly non-negative, the optimal solution set (OSS) of the ILP model is determined as the subscription of the zone created by reversing the signs of the binding constraints of the worst model and the binding constraints of the best model. The solutions to several problems obtained by the new algorithm and a Monte Carlo simulation are compared. The proposed algorithm is applicable to large-scale problems. To this end, an ILP model with 270 constraints and 270 variables is solved.

1. Introduction

Many real world system parameters are inexact and are determined as interval numbers. Therefore, interval linear programming (ILP) is an efficient method to characterize inexact parameters in decision-making problems. Recently, obtaining the optimal solution set (OSS) and the optimal range of the objective values of the ILP problem has become important to researchers. Allahdadi and Mishmast Nehi (2013) determined the OSS of the ILP model from the worst and best model constraints when all of the components of the optimal solutions to the ILP model are strictly non-negative. In fact, by assuming the positivity of all of the components of the feasible point of the best model, if the number of constraints and variables are equal, the exact OSS of the ILP model is parallel to the region that is the subscription of the feasible zone of the best model and the feasible zone created by reversing the inequality signs of the worst model constraints. Several methods have been proposed to solve the ILP model. Some of the methods transform the ILP model into two sub-models (Allahdadi, Mishmast Nehi, Ashayerinasab, & Javanmard, 2016; Fan & Huang, 2012; Huang, Baetz, & Patry, 1995; Huang & Moore, 1993; Lu, Cao, Wang, Fan, & He, 2014; Tong, 1994; Wang & Huang, 2014; Zhou,Huang, Chen, & Guo, 2009). The solution area of these methods is formed by solving these sub-models and obtaining their optimal solutions. One solution, the BWC method (Tong, 1994), obtains the largest interval of the objective function values. In addition, some of these methods, such as a novel ILP (Huang & Moore, 1993), a two-step method (TSM) (Huang et al., 1995), and another solution (SOM-2) (Lu et al., 2014) have been proposed. However, part of the solution area of these methods may be infeasible. Several techniques have been developed to remove the infeasible part of the solution area of these methods, such as the modified ILP (MILP) (Zhou et al., 2009), improved TSM (ITSM) (Wang & Huang, 2014), and robust TSM (RTSM) (Fan & Huang, 2012). Allahdadi et al. (2016), introduced two improvement methods, namely, IMILP and IILP, to delete the non-optimal solutions to the solution areas of the MILP and ILP methods, respectively.

An arbitrary point is a feasible solution to the ILP model if it belongs to the feasible zone of the best model, and it is optimal if it is an optimal solution to an arbitrary characteristic problem of the ILP model. If the optimal solution to a linear programming (LP) model exists, then it lies in the extreme point set or bounds of the feasible region of the LP model. In fact, the optimal solution to the LP problem is the solution to a linear equations system (LES), whose equations are the binding constraints of the LP model. By considering the variables (such as dual variables) corresponding to the binding constraints of the LP model and by multiplying them in the columns of the technological matrix, an LES can be formed

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(similar to the dual problem constraints). In the optimal solution to an LP model, the objective gradient lies in the cone created by the gradients of the binding constraints. This optimality qualification is equivalent to the strong feasibility of the LES presented above. An ILP model is the union of numerous characteristic models, which are LP models. Hence, an interval linear equations system (ILES) can be formed, which is the union of the conventional systems described above.

In this article, the exact OSS of the ILP model will be obtained. A new algorithm is introduced, and an arbitrary characteristic problem of the ILP problem is chosen and solved. The set of indices of its binding constraints is then obtained. With this index set, an ILES is formed and solved using the enclosure method. If all of the components of the interval solutions to this system are strictly nonnegative, the OSS of the ILP model can be obtained.

This algorithm has several advantages. (1) It obtains the exact OSS of the general ILP model when the related ILES is strongly feasible. (2) The solution process of this algorithm is not complex. (3) The algorithm obtains all of the optimal solutions to the ILP model when the solution set of the related ILES is strictly nonnegative, whereas the solution area of previous solution methods only can obtain some results. (4) The OSS obtained in Allahdadi and Mishmast Nehi (2013) is expressed for special cases of the ILP model, but this algorithm obtains the exact OSS of the ILP model in the general case. Hladik (2016) presented a discussion of the topological affections of the robust OSS, and several definitions related to interval numbers and a new method for solving the interval bilevel LP problem were studied by Ren, Wang, and Xue (2017). Several essential definitions and theorems are expressed and proved to illustrate the algorithm described above. In addition, numerical examples are introduced to study the method. Finally, the solutions obtained using the algorithm and the Monte Carlo simulation (MCS) method are compared to illustrate the applicability of the algorithm. A large-scale ILP model with 270 constraints and 270 variables is studied to demonstrate that the proposed algorithm is applicable.

2. Preliminaries

An interval number $a^{\pm}$ is defined as $[a^-, a^+]$, where $a^- \leq a^+$. If $a^- = a^+$, then $a^{\pm}$ will degenerate, and the interval number $a^{\pm}$ transforms into a crisp number.

Definition 2.1. Suppose that, $A^- = (a^-_{ij}),\ A^+ = (a^+_{ij}) \in \mathbb{R}^{m\times n},\ m, n \in \mathbb{N}$ are two matrices that for all $i, j;\ (a^-_{ij}) \leq (a^+_{ij})$. Thus, an interval matrix is given as follows:

$$\mathbf{A}^{\pm} = [A^-, A^+] = \{A \in \mathbb{R}^{m\times n} \mid A^- \leq A \leq A^+\}.$$

The radius and centre of $\mathbf{A}^{\pm}$ are $\Delta_{\mathbf{A}^{\pm}} = \frac{1}{2}(A^+ - A^-)$ and $A^c = \frac{1}{2}(A^- + A^+)$, respectively. Thus, $\mathbf{A}^{\pm} = [A^-, A^+] = [A^c - \Delta_{\mathbf{A}^{\pm}}, A^c + \Delta_{\mathbf{A}^{\pm}}]$. The union of all $m \times n$ interval matrices is denoted by $\mathbb{IR}^{m\times n}$. An interval vector $\mathbf{v}^{\pm}$ is introduced as the set $\mathbf{v}^{\pm} = \{\mathbf{v} \mid \mathbf{v}^- \leq \mathbf{v} \leq \mathbf{v}^+\}$, where $\mathbf{v}^-, \mathbf{v}^+ \in \mathbb{R}^n$ are crisp vectors (Fiedler, Nedoma, Ramik, Rohn, & Zimmermann, 2006).

Definition 2.2. An interval number $a^{\pm}$ is non-negative (positive) if $a^- \geq 0$ ($a^- > 0$) and is non-positive (negative) if $a^+ \leq 0$ ($a^+ < 0$).

A general form of the ILP model is defined as follows:

$$\begin{aligned}
&\text{Minimization } Z^{\pm} = \sum_{j=1}^{n} c_j^{\pm} x_j^{\pm}\\
&\text{subject to: } \sum_{j=1}^{n} a_{ij}^{\pm} x_j^{\pm} \geq b_i^{\pm}, \quad i = 1, \ldots, m,\\
&\qquad\qquad x_j^{\pm} \geq 0, \quad j = 1, \cdots, n.
\end{aligned} \tag{1}$$

In model (1) by considering certain values through interval parameters, a conventional LP model that is called a characteristic model is obtained as follows:

$$\begin{aligned}
&\text{Minimization } Z^{o} = \sum_{j=1}^{n} c_j^{o} x_j\\
&\text{subject to: } \sum_{j=1}^{n} a_{ij}^{o} x_j \geq b_i^{o}, \quad i = 1, \ldots, m,\\
&\qquad\qquad x_j \geq 0, \quad j = 1, \cdots, n.
\end{aligned} \tag{2}$$

where $c_j^o \in c_j^{\pm} = [c_j^-, c_j^+]$, $a_{ij}^o \in a_{ij}^{\pm} = [a_{ij}^-, a_{ij}^+]$, and $b_i^o \in b_i^{\pm} = [b_i^-, b_i^+]$.

Definition 2.3. An interval hyperplane $H^{\pm}$ in $\mathbb{IR}^n$ is defined by the set $\{\mathbf{x}^{\pm} : \mathbf{p}^{\pm}\mathbf{x}^{\pm} = \mathtt{k}^{\pm}\}$, where $\mathbf{p}^{\pm}$ is an interval vector in $\mathbb{IR}^n$, the interval vector $\mathbf{p}^{\pm}$ does not consist of the zero vector, and $k^{\pm}$ is an interval number. A set $H = \{\mathbf{x} : \mathbf{p}\mathbf{x} = \mathtt{k}\} \in H^{\pm}$ where $\mathbf{p} \in \mathbf{p}^{\pm}, \mathtt{k} \in \mathtt{k}^{\pm}$, is called a characteristic hyperplane. An interval hyperplane $H^{\pm}$ is the union of all conventional hyperplanes H.

Definition 2.4. A zone $\hat{X}$ of the feasible region of the ILP model (1) is called an extreme zone if each of its points is an extreme point of a characteristic model of the ILP model (1). In other words, a zone $\hat{X}$ is an extreme zone if it is the intersection of some n linearly independent defining interval hyperplanes of the feasible region of the ILP model (1).

Theorem 2.1. *Tong (1994) In the ILP model (1), the largest and smallest feasible regions are* $\sum_{j=1}^{n} a_{ij}^+ x_j \geq b_i^-, \forall i,\ \ x_j \geq 0,\ \ \forall j$ *and* $\sum_{j=1}^{n} a_{ij}^- x_j \geq b_i^+,\ \forall i,\ x_j \geq 0,\ \forall j$, *respectively.*

Definition 2.5. A point $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)$ is said to be a feasible point of the ILP model (1) if $\sum_{j=1}^{n} a_{ij}^+ \hat{x}_j \geq b_i^-,\ \forall i$, and $\hat{x}_j \geq 0,\ \forall j$.

Theorem 2.2. *Rohn (1993) If the set* $\mathbf{A}\mathbf{x} = \mathbf{b}$, *where* $\mathbf{A} \in \mathbb{IR}^{n\times n}$ *and* $\mathbf{b} \in \mathbb{IR}^n$ *is solved, the interval vector* $\mathbf{r}$ *is obtained by the enclosure method, then for* $i = 1, \cdots, n$:

$$r_i^- = \min\left\{-\mathbf{x}_i^* + (\mathbf{x}_i^c + |\mathbf{x}_i^c|)M_{ii},\ \frac{1}{2M_{ii} - 1}(-\mathbf{x}_i^* + (\mathbf{x}_i^c + |\mathbf{x}_i^c|)M_{ii})\right\},$$

$$r_i^+ = \max\left\{\ \mathbf{x}_i^* + (\mathbf{x}_i^c - |\mathbf{x}_i^c|)M_{ii},\ \frac{1}{2M_{ii} - 1}(\ \mathbf{x}_i^* + (\mathbf{x}_i^c - |\mathbf{x}_i^c|)M_{ii})\right\},$$

where $M = (I - |(A^c)^{-1}|\Delta_{\mathbf{A}})^{-1}$, $\mathbf{x}^c = (A^c)^{-1} b^c$, $\mathbf{x}^* = M(|\mathbf{x}^c| + |(A^c)^{-1}|\Delta_{\mathbf{b}})$, *and* A^c *is a non-singular matrix and* $\rho(|(A^c)^{-1}|\Delta_{\mathbf{A}}) < 1$.

3. Review of previous solution methods

The following subsections present a review of previous solution methods. First, the BWC method is reviewed, and the best and worst models are examined to determine the exact OSS of the ILP model when all of the components of the feasible solution are positive. Then, an introduction to various solution methods and their solution areas is presented.

3.1. BWC method

Tong transformed the ILP problem (1) into worst and best subproblems, which are summarized as follows (Tong, 1994):