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A Linear Programming-based Iterative Approach to Stabilizing Polynomial Dynamics

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Abstract: In this paper, we consider the problem of synthesizing static output feedback controllers for stabilizing polynomial systems. We jointly synthesize a Lyapunov function and a static output feedback controller that stabilizes the system over a given subset of the state-space. Motivated by the numerical issues that are commonly faced using SOS (Sum of Squares)/SDP (Semi-Definite Programming) solvers, we examine a linear programming (LP) based alternative approach that can yield more precise results, in practice. Our approach uses Bernstein polynomials to relax parametric polynomial optimization problems into bilinear optimization problems (BP). Subsequently, we approach the bilinear inequalities using a modified alternating minimization approach that alternates between solving linear programs on complementary sets of variables. Finally, we provide a comparison between our approach and BMI (bilinear matrix inequalities) solvers that tackle the same problem. We conclude that LP/BP relaxation approach is promising and can be more efficient than SDP/BMI relaxations.

Keywords: Bernstein Polynomials, Lyapunov Functions, Stability, Controller synthesis.

1. INTRODUCTION

The problem of designing stabilizing controllers for nonlinear dynamical systems is of great importance. In this paper, we study the problem of synthesizing static output feedback controllers for polynomial systems by solving a polynomial optimization problem to directly obtain the controller along with the associated Lyapunov functions that yields the proof of stability.

Our approach inputs the description of a polynomial system and a desired region R to be stabilized. It then proceeds to find a static output feedback control law and an associated Lyapunov function to ensure local stability in R. We assume a given structure for the feedback as a polynomial function of the outputs of the system. Furthermore, we assume a polynomial template form for the unknown Lyapunov function. We proceed to encode the conditions for the Lyapunov function, obtaining a hard polynomial optimization problem that involves the coefficients of the Lyapunov functions and those of the feedback. Following the LP relaxations presented in Sassi et al. (2016), a so called bilinear optimization problem (BP) is obtained. Then, we iteratively solve this BP problem through an iterative method alternating between linear programs with optimal values getting closer and closer to the solution of the bilinear problem.

The i^{th} iteration of the approach selects a positive definite polynomial V_i and a feedback law u_i. Furthermore, we require V_i' to be negative definite inside the region R for V_i to be a Lyapunov function guaranteeing asymptotic stability. Failing this, we first search for a new positive definite polynomial V_{i+1} whose Lie derivative V_{i+1}' has a larger maxima inside R *fixing* u_i, and adjust to a new feedback law u_{i+1} that improves the maximal value of V_{i+1}' inside R. Each iteration is reduced to solving a Linear Programming (LP) problem obtained using Bernstein polynomials combined with a reformulation linearization technique (RLT) proposed by Sherali and Tuncbilek (1992). It is well-known that iterative approaches does not necessarily converge to a global minimum, in general. However, our evaluation over a wide variety of benchmark examples shows that our approach is effective at converging to a global minimum by discovering an appropriate feedback law u^* and an associated Lyapunov function V^*.

1.1 Related work

Automatic static output feedback design, or more generally, finding feedback that satisfies given structural constraints is well-known to be a hard problem in general. A direct approach given by Henrion et al. (2005) uses the characteristic polynomial of the transfer function matrix, and derives constraints that ensure the Hermite stability criterion for this matrix. As a result, they obtain a system of PMI (polynomial matrix inequalities), that is solved using a local optimization solver (PENBMI). Similarly, the work by Chesi (2014) considers the problem of robust

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output feedback controller synthesis for linear systems that involves the formulation of polynomial constraints using the Routh-Hurwitz criterion. To avoid directly solving bilinear constraints, the approach projects away one set of variables using sum-of-squares by considering a "robust stabilizability" function that is positive over those values of feedback gains that yield a robustly stable closed loop system. In contrast, an indirect approach such as the one proposed here reduces the non convex BMIs to a series of convex LMIs. This was proposed as the $V - K$ iteration by El Ghaoui and Balakrishnan (1994). The approach iteratively solves a bilinear problem by fixing one set of variables while modifying the other to result in a decrease in the objective values. Our goal is to use this technique for polynomial systems while replacing linear matrix inequalities (BMI) and linear matrix inequalities (LMI) with bilinear and linear programs that can be solved more efficiently iteratively. A similar idea for solving bilinear problems appears in the work of Gaubert et al. (2007), for finding invariants for discrete-time systems.

Another class of methods consists on reducing the problem to a set of sum of squares (SOS) formulations (see Zhao and Wang (2010); Nguang et al. (2011) and references therein). In Nguang et al. (2011), an iterative SOS approach is proposed. This approach uses the Schur complement to produce a set of BMIs relaxed to an SOS problem. More precisely, an additional design nonlinear term $\epsilon(x)$ is introduced, and causes bilinearity. An iterative approach is then obtained by fixing a guess for $\epsilon(x)$ and iteratively updating it until feasibility is obtained. Once again, the major problem arises from the fact that the Lyapunov function and a static output feedback are needed simultaneously. Another alternative called diagonally dominant sum-of-squares (DSOS) was proposed by Ahmadi and Majumdar (2014). It proceeds in a similar fashion as the standard relaxation to SDP problems (Parrilo (2003); Shor (1987); Lasserre (2001)). Rather than encoding positive semi-definiteness of the matrices in the resulting SDP, it imposes a stronger criterion of diagonal dominance on these matrices that naturally yields LP relaxation. However, this is still immature and no efficient solvers are developed yet.

Recently, a more related approach was proposed by Ribard et al. (2016) for designing polynomial controllers using Bernstein basis. In this work, only Bernstein coefficients are used and bilinearity is avoided by considering the set of all possible vertices (thanks to the so called convex property of multi-affine functions inside rectangles (Belta and Habets (2006))) . This approach can be expensive since it potentially involves enumerating an exponential number of vertices. It also introduces conservatism since it limits the search to the vertices of a hyper-rectangle rather than a polytope. Note that a similar approach to finding globally optimal solution of bilinear programs was proposed earlier by Floudas and Visweswaran (1990) using ideas from Bender's decomposition.

Finally, other approaches to controlling polynomial systems include the use of nonlinear optimal control techniques such as feedback linearization, back-stepping, and exact linearization. However, those techniques rely on the system being of a certain form, mostly involve state-feedback and are less related to our approach.

1.2 Organization

The paper is organized as follows. In Section 2, we show that solving the stabilization problem is equivalent to search for a polynomial Lyapunov function and an admissible controller inside a polyhedral feasible region. The obtained conditions are then recasted to a parametric polynomial optimization problem (PPOP). In Section 3, we show how those PPOPs can be relaxed to a bilinear feasibility problem. In Section 4, we provide our main algorithm for joint synthesis of polynomial Lyapunov functions and controllers. Finally, numerical results and comparison with SDP based approaches are presented in Section 5.

2. PROBLEM FORMULATION AND POLYNOMIAL OPTIMIZATION PROBLEMS

2.1 Problem formulation

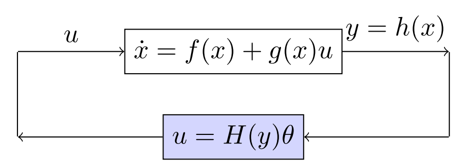

Fig. 1. Overall structure of the controller synthesis problem considered.

In this work, we consider a nonlinear control-affine system subject to input constraints :

$$\begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(y(t)), \ x \in R, \ u \in \mathcal{U}. \\ y(t) = h(x(t)). \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ represents the state variables ranging over an hyper-rectangle R : $[\underline{x_1}, \overline{x_1}] \times \cdots \times [\underline{x_n}, \overline{x_n}]$, $u \in \mathbb{R}^p$ represents the control inputs ranging over a polyhedral set $\mathcal{U} = \{u \in \mathbb{R}^p | \ \alpha_{\mathcal{U},k} \cdot u \leq \beta_{\mathcal{U},k}, \ \forall k \in \mathcal{K}_{\mathcal{U}}\}$ where $\alpha_{\mathcal{U},k} \in \mathbb{R}^p$, $\beta_{\mathcal{U},k} \in \mathbb{R}$, $\mathcal{K}_{\mathcal{U}}$ is a finite set of indices, and $y \in \mathbb{R}^q$ are the outputs.

We assume that the functions $f : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}^q$ and the control matrix $g : \mathbb{R}^n \to \mathbb{R}^{(n \times p)}$ defining the dynamics of the system are multivariate polynomial maps and that $x^* = 0_n$ is an equilibrium for the system (1), i.e $f(0_n) + g(0_n)u(h(0_n)) = 0_n$.

The choice of rectangular region R and polyhedral region $\mathcal{U}$ is motivated by the linear relaxation that will be used later, and by the fact that we will deal with local stability. We usually pick $R = [-1, 1]^n$ as a reasonable stabilizing region that can be varied for each example using a scaling factor).

Stabilizing Feedback: In this work, we assume that the desired feedback is given by a function $u : \mathbb{R}^q \to \mathcal{U}$ mapping outputs y to control inputs u to yield a closed-loop system

$$\dot{x} = f(x) + g(x)u(y), \quad y = h(x) \tag{2}$$

We require that the closed loop system (2) be asymptotically stable in R. This is achieved by ensuring the existence of a local differentiable Lyapunov function. More precisely, the system (2) is asymptotically stable in R if there exist a function $V(x)$ such that