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A double-sided dynamic programming approach to the minimum time problem and its numerical approximation

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ABSTRACT

We introduce a new formulation of the minimum time problem in which we employ the *signed* minimum time function positive outside of the target, negative in its interior and zero on its boundary. Under some standard assumptions, we prove the so called *Bridge Dynamic Programming Principle* (BDPP) which is a relation between the value functions defined on the complement of the target and in its interior. Then owing to BDPP, we obtain the error estimates of a semi-Lagrangian discretization of the resulting Hamilton–Jacobi–Bellman equation. In the end, we provide numerical tests and error comparisons which show that the new approach can lead to significantly reduced numerical errors.

1. Introduction

The global solution of the minimum time problem can be efficiently obtained via the solution of the associated Hamilton–Jacobi–Bellman equation. Indeed, the unique viscosity solution of this equation is the optimal value function of the problem, whose knowledge can in a subsequent step be used in order to synthesize the optimal control functions. For the numerical solution of this Hamilton–Jacobi–Bellman equation, semi-Lagrangian schemes — which consist of a semi-discretization in time followed by a finite element discretization in space — are particularly attractive because they are unconditionally stable and allow to combine different discretization methods in space and time [7]. Most importantly, however, the semi-discretization is directly linked to a discrete time approximation to the original minimum time problem, which facilitates both the interpretation of the numerical results and the synthesis of approximately optimal feedback laws from the numerical approximation.

In the case of the minimum time problem, the semi-Lagrangian approach was first presented and analyzed in [2,3], recent developments include the analysis of high-order discretization schemes in time in [4]. One of the main disadvantages of the semi-Lagrangian approach is the fact that the semi-discretization of the standard minimum time problem leads to a piecewise constant optimal value function whose discontinuities pose problems, e.g., for the subsequent spatial discretization. The discontinuities stem from the fact that the optimal value function is fixed to $v \equiv 0$ on the target set of the minimum time problem. In order to improve the approximation, it does hence appear to be a good idea to use a formulation of the minimum time problem which avoids setting v to 0. To this aim, we extend the original problem by introducing an

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additional problem defined in the interior of the target. While this requires somewhat more regularity of the target boundary and the vector field, we demonstrate in this paper that it also yields a significant improvement in accuracy. We present the theoretical foundations of this new formulation of the minimum time problem as well as its numerical discretization including an error analysis of the resulting semi-Lagrangian scheme. Numerical examples show that under suitable conditions the new formulation is indeed able to significantly reduce the numerical error compared to the classical approach.

The remainder of the paper is organized as follows. Section 2 introduces the new formulation of the minimum time problem from a theoretical point of view and proves the bridge dynamic programming principle (BDPP) as the main technical tool for the subsequent analysis. In Section 3 the discretization is introduced and the numerical error is analyzed. The performance of the new approach is finally illustrated by several numerical examples in Section 4.

2. A new formulation of the minimum time problem

To begin with, we recall basic notations and definitions necessary for this work. Let $S \subset \mathbb{R}^n$ be a closed set and $\sigma > 0$ be a given constant. For $x \in \mathbb{R}^n$, we define

$$d_S(x) = \min\{\|y - x\| : y \in S\},$$
$$S_\sigma = \{x \in \mathbb{R}^n : d_S(x) \le \sigma\},$$
$$S_{-\sigma} = \{x \in \mathbb{R}^n : d_{\overline{S^c}}(x) \ge \sigma\},$$

where S^c is the complement of S, $S^c = \mathbb{R}^n \setminus S$, and $\bar{S}$ is the closure of S. For readers' convenience, we also recall the definition of a set satisfying an internal (external) sphere condition as follows.

Definition 2.1. Let $S \in \mathbb{R}^n$ be closed and $\rho > 0$ be given.

(1) S satisfies a ρ-internal sphere condition if S is the union of closed spheres of radius ρ, i.e., for any $x \in S$ there exists y such that $x \in \overline{B_\rho(y)} \subset S$.
(2) S satisfies a ρ-external sphere condition if $\bar{S}^c$ satisfies a ρ-internal sphere condition.

Consider the control dynamics and its inverse one in $\mathbb{R}^n$

$$\begin{cases} \dot{y}^+(t) &= f(y^+(t), u(t)) \\ y(0) &= \xi, \end{cases} \tag{2.1}$$

$$\begin{cases} \dot{y}^-(t) &= -f(y^-(t), u(t)) \\ y(0) &= \xi, \end{cases} \tag{2.2}$$

where $u(t) \in U$ for a. e. t, $U \subset \mathbb{R}^m$ a compact set. Under standard assumptions, the existence and uniqueness of (2.1) as well as (2.2) are guaranteed for any $u(\cdot)$ measurable and any $\xi \in \mathbb{R}^n$. Let $S \subset \mathbb{R}^n$, a nonempty compact set, be the target and $\mathcal{U} := \{u : [0, +\infty) \to U, \textit{measurable}\}$. We define the minimum time to reach S and to $\bar{S}^c$ by following some $u(\cdot) \in \mathcal{U}$ from $\xi \notin S$ and $\eta \notin \bar{S}^c$ respectively

$$t_S(\xi, u) = \min\{t : y^+(t, \xi, u) \in S\} \le +\infty,$$
$$t_{S^c}(\eta, u) = \min\{t : y^-(t, \eta, u) \in \bar{S}^c\} \le +\infty.$$

Then the minimum time functions to reach S and to reach $\bar{S}^c$ from ξ and η are defined respectively as

$$T_S(\xi) = \inf_{u \in \mathcal{U}} \{t_S(\xi, u)\},$$
$$T_{S^c}(\eta) = \inf_{u \in \mathcal{U}} \{t_{S^c}(\eta, u)\}.$$

Under standard assumptions, the infimum is attained, provided it is not $+\infty$. We also define

$$\mathcal{R}^S = \{\xi \in \mathbb{R}^n : T_S(\xi) < +\infty\},$$
$$\mathcal{R}^{S^c} = \{\eta \in \mathbb{R}^n : T_{S^c}(\eta) < +\infty\},$$

the reachable sets w.r.t S and $\bar{S}^c$. We define level sets in a neighborhood of ∂S by setting, given $\tau > 0$,

$$S_\tau^+ = \{x \notin S,\ T_S(x) < \tau\},$$
$$S_\tau^- = \{x \notin \bar{S}^c,\ T_{S^c}(x) < \tau\}.$$