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Calculation of maximum entropy densities with application to income distribution

Ximing Wu*

Department of Agricultural and Resource Economics, University of California at Berkeley, Berkeley, CA 94720, USA, and Department of Economics, University of Guelph, 'Guelph, Ont., Canada

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Abstract

The maximum entropy approach is a flexible and powerful tool for density approximation. This paper proposes a sequential updating method to calculate the maximum entropy density subject to known moment constraints. Instead of imposing the moment constraints simultaneously, the sequential updating method incorporates the moment constraints into the calculation from lower to higher moments and updates the density estimates sequentially. The proposed method is employed to approximate the size distribution of U.S. family income. Empirical evidence demonstrates the efficiency of this method.

JEL classification: C4; C6; D3

Keywords: Maximum entropy; Density estimation; Sequential updating; Income distribution

0. Introduction

A maximum entropy (maxent) density can be obtained by maximizing Shannon's information entropy measure subject to known moment constraints. According to Jaynes (1957), the maximum entropy distribution is "uniquely determined as the one which is maximally noncommittal with regard to missing information, and that it agrees with what is known, but expresses maximum uncertainty with respect to all other matters."

The maxent approach is a flexible and powerful tool for density approximation, which nests a whole family of generalized exponential distributions, including the exponential, Pareto, normal, lognormal, gamma, beta distribution as special cases.

* Tel.: +1-510-642-8179; fax: +1-510-643-8911.
E-mail address: ximing@are.berkeley.edu (X. Wu).

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The maxent density has found some applications in econometrics. For example, see Zellner (1997) and Zellner and Tobias (2001) for the Bayesian method of moments, which uses the maxent technique to estimate the posterior density of parameters of interest; and Buchen and Kelly (1996), Stutzer (1996) and Hawkins (1997) for some applications in finance.

Despite its versatility and flexibility, the maxent density has not been widely used in empirical studies. One possible reason is that there is generally no analytical solution for the maxent density problem and the numerical estimation is rather involved. In this study, I propose a sequential updating method for the calculation of maxent densities. Compared to the existing studies that consider the estimation of the maxent density subject to just a few moment constraints, the proposed method is able to calculate the maxent density associated with a much higher number of moment constraints. This method is used to approximate the size distribution of U.S. family income distribution.

1. The maxent density

The maxent density is typically obtained by maximizing Shannon's entropy (defined relative to uniform measure),

$$W = \int -p(x)\log p(x)\,\mathrm{d}x,$$

subject to some known moment constraints or equations of moments.[1] Following Zellner and Highfield (1988), Ormoneit and White (1999), and Rockinger and Jondeau (2002), we consider only the arithmetic moments of the form

$$\int x^i p(x)\,\mathrm{d}x = \mu_i, \qquad i = 0, 1, \ldots, k. \tag{1}$$

Extension to more general moments (e.g., the geometric moments, $E(\ln^i x)$ for $x > 0$) is straightforward (Soofi et al., 1995; Zellner and Tobias, 2001).

We use Lagrange's method to solve for the maxent density. The solution takes the form

$$p(x) = \exp\left(-\sum_{i=0}^{k} \lambda_i x^i\right), \tag{2}$$

where λ_i is the Lagrangian multiplier for the ith moment constraint. Since an analytical solution does not exist for $k \geqslant 2$, one must use a nonlinear optimization technique to solve for the maxent density. One way to solve the maxent problem is to transform the constrained optimization problem into an unconstrained optimization problem using the dual approach (Golan et al., 1996). Substituting Eq. (2) into the Lagrangian function and rearranging terms, we have the dual objective function for an unconstrained

[1] Mead and Papanicolaou (1984) give the necessary and sufficient condition for the moments that lead to a unique maxent density. We find that the sample moments of any finite sample satisfy this condition. The proof is available from the author upon request.