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Bifurcation analysis of an enzyme-catalyzed reaction–diffusion system

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ABSTRACT

In this paper, we concern about the dynamics of a diffusive enzyme-catalyzed system arising from glycolysis, describing a biochemical reaction in which a substrate is converted into a product with positive feedback and into a branched sink. The temporal and spatiotemporal dynamics of the system under homogeneous Neumann boundary conditions, are studied. Preliminary analysis on the local asymptotic stability and Hopf bifurcation of the spatially homogeneous model based on ordinary differential equation is presented. For the reaction–diffusion model, firstly the parameter regions for the stability or instability of the unique constant steady state are discussed. Finally, bifurcations of spatially homogeneous and nonhomogeneous periodic solutions as well as nonconstant steady state solutions are studied. Numerical simulations are presented to verify and illustrate the theoretical results.

1. Introduction

Natural systems exhibit an amazing diversity of structures in both living and non-living mechanisms, and thereby, in-depth understanding of spatial and temporal behavior of interacting species or reactants in ecological and chemical dynamics has become a central issue. For this purpose, numerous coupled partial differential equations have been proposed by biologists, chemists and applied mathematicians to model problems arising from various disciplines such as population dynamics, genetics and chemical reactions [1].

In the early 1950s, the British mathematician Alan M. Turing [2] proposed a model that accounts for pattern formation in morphogenesis. Turing showed mathematically that a system of coupled reaction–diffusion equations could give rise to spatial concentration patterns of a fixed characteristic length from an arbitrary initial congratulation due to so called diffusion-driven instability, that is, diffusion could destabilize an otherwise stable equilibrium of the reaction–diffusion system and lead to nonuniform spatial patterns. Turing's analysis stimulated considerable theoretical research on mathematical models of pattern formation, and a great deal of research have been devoted to the study of Turing instability in chemical and biology contexts, see for example, [3–9].

This paper is concerned with a biochemical reaction–diffusion scheme comprising a substrate (S), a product (P), and three enzymes (E_i, $i = 1, 2, 3$) in which a substrate is converted to a product with positive feedback and into a branched sink. The reaction process of the model is given by [10–12]

$$\xrightarrow{v_0(\text{input})} S \xrightarrow{E_3(\text{sink})},$$
$$S \xrightarrow{E_1(\text{conversion})} P \xrightarrow{(\text{feedback})} E_1,$$
$$P \xrightarrow{E_2(\text{sink})}.$$

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The corresponding system of ODEs is as follows:

$$\begin{aligned}\frac{ds}{dt} &= \nu_0 - V_{E_1} - V_{E_3},\\ \frac{dp}{dt} &= V_{E_1} - V_{E_2} \quad \text{in } (0,\infty),\end{aligned}$$

where s and p are the concentrations of the substrate and product, respectively; ν_0 (input) is a positive constant; and the reaction rates V_{E_i} $(i = 1,2,3)$ are functions of s and p, which will be determined concretely below. Many such reactions are present in actual biochemical systems with the specific example dealt with in [13] being related to glycolysis in yeast cells [14,11]. The term V_{E_3} plays an essential role in obtaining the formation and disappearance of large periodic solutions [11]. In our study, this interesting reaction rate is considered to be of the basic form $V_{E_3} = \nu_2 s$, for a positive constant ν_2, as described in [10,14], and a rescaled value of ν_2 is considered to be the main parameter in the current study; The rate V_{E_1} is considered to be $V_{E_1} = \nu_1 sp$, for a positive rate constant ν_1 (e.g. see [10,12]).

The most well-known feature of enzymatic reactions is that an enzyme can be saturated. With the saturation of an enzyme, the rate of the reaction catalyzed by the enzyme reaches its maximum, and a further increase in the substrate concentration does not increase the reaction rate. This feature was first observed by Michaelis and Menten. Thus, it is reasonable that E_2 in the third reaction follows the Michaelis–Menten kinetics, that is,

$$V_{E_2} = \frac{\nu_3 p}{p+\nu_4},$$

for a positive rate constant ν_3 and a positive Michaelis constant ν_4 [10,12,13].

By substituting the above reaction rates V_{E_i}, $(i = 1,2,3)$ into the ordinary differential equations (ODEs), we can eventually obtain the autocatalytic differential system:

$$\begin{aligned}\frac{ds}{dt} &= \nu_0 - \nu_1 sp - \nu_2 s,\\ \frac{dp}{dt} &= \nu_1 sp - \frac{\nu_3 p}{p+\nu_4} \quad \text{in } (0,\infty).\end{aligned}$$

In the above ODEs, when this effect is considered (e.g. in living cells), the concentrations involved in the reaction process are spatially dependent, hence, the equations governing these concentrations become partial differential equations (PDEs):

$$\begin{aligned}\frac{\partial s(t,x)}{\partial t} &= \nu_0 - \nu_1 s(t,x)p(t,x) - \nu_2 s(t,x) + D_1 \Delta s(t,x),\\ \frac{\partial p(t,x)}{\partial t} &= \nu_1 s(t,x)p(t,x) - \frac{\nu_3 p(t,x)}{p(t,x)+\nu_4} + D_2 \Delta p(t,x) \quad \text{in } (0,\infty)\times\Omega,\end{aligned}$$

where D_1 and D_2 are positive constants, and $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$. In this study, we deal with PDEs under the homogeneous Neumann boundary conditions.

For simplicity, after introducing L as a typical length scale and defining the following new dimensionless quantities [12],

$$\begin{aligned}&\bar{t} = \frac{\nu_3}{\nu_4}t,\ \bar{x} = \frac{x}{L},\ u = \frac{\nu_3}{\nu_0\nu_4}s,\ v = \frac{\nu_1\nu_4}{\nu_3}p,\ \delta = \frac{\nu_2\nu_4}{\nu_3},\\ &\lambda = \frac{\nu_0\nu_1\nu_4^2}{\nu_3^2},\ \mu = \frac{\nu_3}{\nu_1\nu_4^2},\ d_1 = \frac{\nu_4}{\nu_3}\frac{D_1}{L^2},\ d_2 = \frac{\nu_4}{\nu_3}\frac{D_2}{L^2},\end{aligned}$$

and then ignoring the upper bars and still denoting the rescaled spatial region by Ω, we can rewrite the specific system in the following dimensionless form:

$$\begin{aligned}\frac{\partial u}{\partial t} &= 1 - uv - \delta u + d_1 \Delta u,\\ \frac{\partial v}{\partial t} &= \lambda uv - \frac{v}{\mu v + 1} + d_2 \Delta v \quad \text{in } (0,\infty)\times\Omega.\end{aligned} \tag{1}$$

In this study, we deal with (1) under the homogeneous Neumann boundary conditions:

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \ \text{ for } x \in \partial\Omega,\ \ t > 0, \tag{2}$$

where ν is the outward normal to Ω at $x \in \partial\Omega$. The homogeneity, corresponds to the "no flux condition", that is no individual can leave or enter the domain via the boundaries.

An important concern is the existence of positive steady-state solutions and the asymptotic behavior of time-dependent solutions. For previous background on stationary patterns and the existence and nonexistence of nonconstant positive solutions of (1), we refer the readers to [12,15,1] and references therein.

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