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Closed-loop stability analysis of discrete-time negative imaginary systems[☆]

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ABSTRACT

We derive necessary and sufficient conditions for stability analysis of a positive feedback interconnection of a discrete-time negative imaginary system and a discrete-time strictly negative imaginary system. General stability analysis results for continuous-time negative imaginary systems connected in positive feedback have recently been proposed. Those recent results extend previous theorems by removing restrictive assumptions on the infinite frequency gains imposed in the earlier literature and by extending the class of negative imaginary systems for which the results are applicable to include systems with free body dynamics (i.e., poles at the origin). Here, we present the discrete-time counterparts of the aforementioned recently developed results which specialise to simple and easy-to-check conditions under specific assumptions. Last, we illustrate some of the results by several examples.

1. Introduction

Negative imaginary systems theory is emerging as a powerful complement to positive real theory and passivity theory. The negative imaginary systems class was first studied in [1]. Negative imaginary systems arise in a wide variety of applications, including nano-positioning systems [2–5], multi-agent systems [6,7], lightly damped structure [8–10], vehicle platoons [11], etc. A rich sequence of results has also appeared in the theory of negative imaginary systems in recent years, including extensions to Hamiltonian systems [12], non-rational systems [13–16], non-proper systems [13,17], infinite-dimensional systems [18], descriptor systems [19], strongly strict negative-imaginary systems [20] and controller synthesis for negative imaginary systems [21–24]. According to [25,26], some possible future work for filtering problems could be further developed.

Stability analysis results of positive feedback interconnections of negative imaginary systems play a central role in negative imaginary systems theory. [1] proposed that, under assumptions on the gains of systems at infinite frequency, a necessary and sufficient condition for the internal stability of a positive feedback interconnection of negative imaginary systems can be expressed as a one-sided restriction on the dc loop gain. This stability result was shown to hold true even for negative imaginary systems with poles on the imaginary axis [27]. These key results have subsequently been developed further to allow negative imaginary systems to have possible poles at the origin [28]. [29] then sought to remove the assumptions on the infinite frequency gains, i.e., $M(\infty)N(\infty) = 0$ and $N(\infty) \geqslant 0$, by using integral quadratic constraint theory and derived sufficient conditions (which are not necessary) for stability analysis. Necessary and sufficient conditions that remove the assumptions on the infinity frequency gains were recently derived in [30]. In contrast with complicated matrix factorisations used in [28] which loose intuition and restrict the applicability of the results and in contrast with sufficiently only conditions developed in [29], a linear shift transformation technique is used in [30] to establish general necessary and sufficient stability analysis results applicable for the full class of negative imaginary systems including those with free body dynamics (i.e., poles at the origin).

The above theory has all been developed in continuous-time. The notion of a discrete-time negative imaginary systems was proposed in [14,31] to fill the gap in the literature. By using a bilinear transformation, a discrete-time negative imaginary lemma was derived, in terms of a discrete-time state-space representation, to characterise discrete-time negative imaginary systems [14,31]. Furthermore, it was shown in [14] that the stability of discrete-time negative imaginary systems only depends on gains at $z = +1$ under specific assumptions analogous to the early assumptions in continuous-time. Here, we extend the stability theorem proposed in [14] for the full class of real, rational, proper discrete-time negative imaginary systems available in the literature without imposing the restrictive assumptions, i.e., $P(-1)Q(-1) = 0$ and $Q(-1) \geqslant 0$. The results in this paper can be considered as, not

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only generalisations of previous work [14,31] but also, discrete-time counterparts of the general continuous-time results in [30].

In this paper, we first state the definitions of discrete-time negative imaginary systems. We then remove two restrictive assumptions in the existing literature, i.e., $P(-1)Q(-1) = 0$ and $Q(-1) \geqslant 0$, imposed in [14] and subsequently derive necessary and sufficient conditions for internal stability of a discrete-time negative imaginary system without poles at $z = +1$ and $z = -1$ connected in positive feedback with a discrete-time strictly negative imaginary system. Then, these results are extended to the case where a discrete-time negative imaginary system with possible poles at $z = +1$ is connected in positive feedback with a discrete-time strictly negative imaginary system. Furthermore, we specialise these general stability theorems in the single-input single-output (SISO) setting to reveal simple and intuitive tests. Additional multiple-input multiple-output (MIMO) specialisations are also given as corollaries to give simple and elegant tests for checking feedback stability. Stability conditions with or without a loop-shifting matrix Ψ are also presented for determining the internal stability of discrete-time negative imaginary systems connected in positive feedback. Lastly, two examples are given to illustrate the importance of some of the results.

Notation: $\Re(a)$ represents the real part of a complex number a. $\bar{\lambda}(A)$ [respectively, $\underline{\lambda}(A)$] denote the largest [respectively, smallest] eigenvalue of a square complex matrix A that has only real eigenvalues. A^* and A^T denote the complex conjugate transpose and transpose of a complex matrix A respectively. $[P(z), Q(z)]$ denotes the positive feedback interconnection of $P(z)$ and $Q(z)$. I_m denotes an identity matrix with dimensions m by m.

2. Preliminaries

We first recall the notion of a discrete-time negative imaginary system with possible poles at $z = +1$.

Definition 1 (*[14,31]*)**.** Let $R(z)$ be a discrete-time, real, rational, proper transfer function. Then, $R(z)$ is said to be Discrete-Time Negative Imaginary (D-NI) if

(1) $R(z)$ has no poles in $\{z \in \mathbb{C} : |z| > 1\}$;
(2) $j[R(e^{j\theta}) - R(e^{j\theta})^*] \geqslant 0$ for all $\theta \in (0, \pi)$ except the values of θ where $z = e^{j\theta}$ is a pole of $R(z)$;
(3) if $z_0 = e^{j\theta_0}$ with $\theta_0 \in (0, \pi)$ is a pole of $R(z)$, then it is a simple pole and the residue matrix $K_0 = {z_0}^{-1}\lim_{z\to z_0}(z - z_0)jR(z)$ is Hermitian and positive semidefinite;
(4) if $z = 1$ is a pole of $R(z)$, then $\lim_{z\to 1}(z - 1)^k R(z) = 0$ for all integer $k \geqslant 3$ and $\lim_{z\to 1}(z - 1)^2 R(z)$ is Hermitian and positive semidefinite;
(5) if $z = -1$ is a pole of $R(z)$, then $\lim_{z\to -1}(z + 1)^k R(z) = 0$ for all integer $k \geqslant 3$ and $\lim_{z\to -1}(z + 1)^2 R(z)$ is Hermitian and negative semidefinite.

[14] considers non-rational systems. To handle possibly non-rational systems, [14] imposes a symmetric assumption. As stated in Remark 3.2 of [14], when one restricts attention to rational systems (as we do in this paper), the symmetric assumption is no longer needed. The five conditions in Lemma 3.2 of [14] with the condition corresponding to symmetry removed, are hence used to directly define rational discrete-time systems as in Definition 1. This definition is also identical to that used in [31].

The following definition describes discrete-time strictly negative imaginary systems.

Definition 2 (*[14]*)**.** Let $R(z)$ be a discrete-time, real, rational, proper transfer function. Then, $R(z)$ is said to be Discrete-Time Strictly Negative Imaginary (D-SNI) if

(1) $R(z)$ has no poles in $\{z \in \mathbb{C} : |z| \geqslant 1\}$;
(2) $j[R(e^{j\theta}) - R(e^{j\theta})^*] > 0$ for all $\theta \in (0, \pi)$.

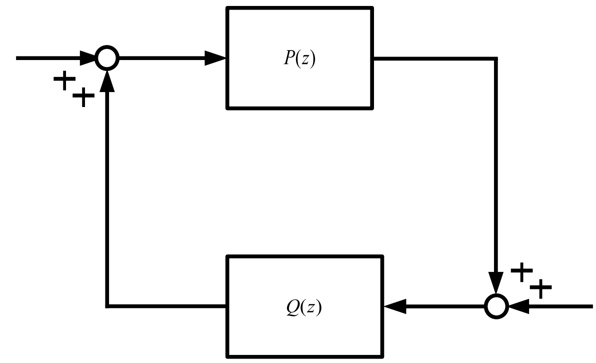

Fig. 1. Positive feedback interconnection of $P(z)$ and $Q(z)$.

3. Main results, part 1: no poles at +1 and −1

In [30], necessary and sufficient conditions for checking the internal stability of a positive feedback interconnection of a *continuous-time*, proper, negative imaginary system without poles at the origin and a *continuous-time* strictly negative imaginary system were derived. The necessary and sufficient conditions in [30] generalised the original result in [1] by removing restrictive assumptions on the infinite frequency gains of the two systems. In this section, we consider the case where a *discrete-time* negative imaginary system and a *discrete-time* strictly negative imaginary system are interconnected via positive feedback as shown in Fig. 1. We hence introduce discrete-time feedback stability theorems that remove restrictive assumptions imposed in earlier literature (e.g., [14]). These results are applicable for negative imaginary systems without poles at $z = +1$ and $z = -1$ and they are hence discrete-time counterparts of the work in Section 3 of [30].

Theorem 3. *Let $P(z)$ be a discrete-time, real, rational, proper, negative imaginary system without poles at $z = +1$ and $z = -1$, and let $Q(z)$ be a discrete-time, real, rational, proper, strictly negative imaginary system. Then, $[P(z), Q(z)]$ is internally stable if and only if*

$I - P(-1)Q(-1)$ is nonsingular,
$\bar{\lambda}[[I - P(-1)Q(-1)]^{-1}(P(-1)Q(1) - I)] < 0$, and
$\bar{\lambda}[[I - Q(1)P(-1)]^{-1}(Q(1)P(1) - I)] < 0.$

Proof. Let $M(s) = P((1+s)/(1-s))$ and $N(s) = Q((1+s)/(1-s))$ via the bilinear transformation $z = (1+s)/(1-s)$. Then, the result follows from [30, Theorem 9]. □

Note that the inequality conditions in Theorem 3 (and indeed in all negative imaginary results) are one-sided restrictions because the maximum eigenvalue of matrices, that have only real eigenvalues, can be either positive or negative.

Also, note that no poles at $z = +1$ in discrete-time corresponds to no poles at the origin in continuous-time, whereas no poles at $z = -1$ in discrete-time corresponds to no poles at infinite frequency (i.e., a proper system) in continuous-time.

Theorem 3 removes the assumptions imposed in [14, Theorem 4.1], i.e., $Q(-1) \geqslant 0$ and $P(-1)Q(-1) = 0$, and as a consequence generalises that result.

The following example is used to demonstrate the usefulness of the result stated in Theorem 3.

Example 1. Consider a positive feedback interconnection of $P(z)$ and $Q(z)$ as shown in Fig. 1 where

$$P(z) = \begin{bmatrix} \dfrac{-z+1}{6z+4} & \dfrac{-5z-5}{6z+4} \\ \dfrac{-9z^2-10z-1}{12z^2+8z} & \dfrac{15z^2+32z+13}{12z^2+8z} \end{bmatrix}$$