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Efficient belief propagation in second-order Bayesian network for singly-connected graphs

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ABSTRACT

Second-order Bayesian networks extend Bayesian networks by incorporating uncertainty in the conditional probabilities. This paper develops a method for inference in a binary second-order Bayesian network with a singly-connected graph that builds upon the message-passing algorithm for regular belief propagation by leveraging recent developments in subjective logic. The method applies the moment-matching approach to the Beta representation of the uncertain probabilities. We provide experimental analysis which shows that the introduced method effectively captures the bounds for the actual error in a consistent manner and, at the same time, does not decrease the efficiency of the performance compared to the other similar approaches.

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1. Introduction

Effective decision making requires the understanding of the probability distribution of different outcomes to compute an expected utility [1]. However, the understanding of the outcome distribution must be learned from relevant prior experiences so that one can determine the distribution of unknown quantities in light of other observed quantities. These prior experiences can be subjective in that a subject matter expert uses his/her recollection of experiences in past situations to determine the probability of various outcomes. Alternatively, these prior experiences can be objective in that past outcomes have been recorded to determine the outcome probabilities. In many business scenarios that are trying to predict the behaviors of consumers, the availability of big data enables reasonable characterization of the outcome probabilities. However in many scenarios dealing with adversarial behaviors such as military campaigns and counterterrorism operations, training data is sparse and/or subject matter experts have limited experience to elicit the probabilities. As a result, these probabilities are highly uncertain and traditional decision making aids are unable to capture the risk associated to operating under uncertainty. Recent efforts are investigating decision making in environments with highly uncertain probabilities [2,3].

More precisely, decision making requires the integration of evidence from multiple confounding factors that determine the likelihood of the possible outcomes. It is common to represent these confounding factors by a probabilistic graphical model [4]. Effective decision making then becomes the determination of the outcome probabilities through inference over the observed factors (or variables). The prior experiences determine the statistical correlations between the variables at decision making time using past instantiations of the values of these variables. Many probabilistic graphical models exist including Bayesian networks (BNs), Markov random fields, factor graphs, etc. BNs [5] have proven to be an effective tool for

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probabilistic reasoning. However, they require significant amounts of prior experiences, subjective and/or objective, to elicit the necessary input conditional probabilities. Uncertainty about probabilities is known as second-order uncertainty [6], and standard BNs processing is unable to accommodate such uncertainty.

There are a number of theories for reasoning under uncertainty trying to extend Bayesian reasoning, with the theory of imprecise probabilities [7], credal networks [8], and the belief theory of evidence [9] being some of the more established ones. More recently, subjective logic (SL) was introduced to connect uncertain probabilistic reasoning with second-order Bayes [10,11]. Specifically, SL operates with subjective opinions which map to Dirichlet distributions for the probabilities. A subjective opinion is a belief-and-uncertainty representation of an unknown probability distribution of a random variable, which together with an assumed prior distribution projects into an estimate for the actual probability distribution of the variable. This estimate equals the mean of the corresponding Dirichlet distribution.

The operators within SL are mostly designed using this connection to Dirichlet distributions as a guiding principle. Some of the operators such as deduction, however, make looser connections to the Dirichlet distribution and employ an uncertainty maximization principle to limit the belief commitments [12,13]. Alternative operators that select the uncertainty to match the variance of the Dirichlet distribution have been proposed in [14].

Recently, subjective Bayesian networks were introduced to extend BNs such that the conditional probabilities become conditional subjective opinions [15,16]. There are two major approaches towards inference in subjective Bayesian networks. The first performs inference composing traditional SL operators, and an overview of this approach is discussed in [11]. The work in this paper belongs to the second approach that makes use of the equivalent Dirichlet representation of subjective opinions and applies the moment matching method to determine the uncertainty. An initial example of the latter approach was presented for a three-node network in [16]. [17] expands on that approach by developing a more general subjective belief propagation method that is able to determine the marginal subjective opinions for any variable in a subjective Bayesian network with a *tree* structure. In essence, inference over subjective Bayesian networks considers second-order Bayesian reasoning where the conditional probabilities are random variables drawn from Dirichlet distributions. In the sequel, we refer to subjective Bayesian networks instead as second-order Bayesian networks (SOBNs) to clarify this distinction.

This paper generalizes further the second approach developing a second-order belief propagation method for reasoning in a SOBN with a singly-connected graph. The method generalizes Pearl's message-passing algorithm for regular belief propagation (BP) from [18] to incorporate uncertainty about the probabilities in a principled manner. We restrict our attention to binary random variables in which case the corresponding Dirichlet representation becomes a Beta distribution over one of the probability values. Then the probability messages between the nodes are shared in the form of Beta distributions and the reasoning through the moment matching method determines a Beta distribution of the result by matching the mean and variance of the actual distribution.

In other words, this work considers a set of binary variables whose influence on each other is represented by a ground truth BN. The structure of the network is assumed to be known, but the conditional probabilities must be learned by observing the joint values of variables and their parents over a number of independent instantiations of the network variables. In light of a uniform prior for the conditional probabilities, the observation of the Bernoulli trials lead to uncertain conditional probabilities whose values are known within Beta distributions. Given these second-order conditional probabilities and a set of the observed variables as evidence, this papers develops an efficient inference method to determine the second-order marginal probabilities of all the unobserved variables conditioned on the evidence. Each marginal probability is a query that is actually a random variable, and the inference method characterizes both the query value and its spread as a Beta distribution. The second-order probabilities should accurately reflect the uncertainty of the query answer in light of the limited training examples to learn an estimate for the parameters of the BN, e.g., the conditional probabilities. In essence, this paper provides an efficient method for second order Bayesian reasoning over uncertain BNs.

Other methods do exist that incorporate uncertainty with reasoning over probabilistic graphical models. For instance, the valuation-based system (VBS) provides a framework to incorporate extensions of Bayes' rule using various uncertainty theories [19]. However, the framework does not connect the uncertainties to second-order Bayes, which is what makes reasoning over SOBNs attractive. $\mathcal{S}$DL-Lite is a recent framework that combines subjective logic opinions with description logic to reason over uncertain probabilistic graphs [20]. It develops a subjective semantic interpretation of the DL-Lite operators and finds the most general opinions to satisfy this semantics. While $\mathcal{S}$DL-Lite considers the connection of subjective logic opinions and evidence to build up a Dirichlet distribution, it does not explicitly incorporate the properties of the Dirichlet distribution to define the semantic interpretation of the logical operators. Credal networks [8] are extensions of BNs where the marginal and conditional probabilities belong to a convex set of probability distributions generated by linear constrains. In the case of binary random variables, credal sets coincide with a set of intervals, i.e. the probability values belong to closed intervals. The 2U algorithm extends the message-passing approach for the latter case of credal networks [21]. However, the 2U algorithm is not designed to capture second order knowledge of the distribution of the conditionals. Rather, it assumes knowledge of the lower and upper bounds for the expected values of the conditional probabilities. There has been work on reasoning in uncertain probabilistic graphs also in the framework of belief theory and the transferable belief model [22]. The belief propagation in belief networks that have Dempster–Shafer belief functions instead of probability functions associated with each node, is done in [22] by applying the generalized Bayes' theorem and the disjunctive rule of combination. Again, these methods do not connect the uncertainties to second-order Bayes.

It should be noted that Beta (and more generally Dirichlet) distributions are commonly used in learning the parameters for discrete variables in BNs. Latent variable models virtually create additional latent variables whose values are parameters