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Better response dynamics and Nash equilibrium in discontinuous games

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ABSTRACT

Philip Reny's approach to games with discontinuous utility functions can work outside its original context. The existence of Nash equilibrium and the possibility to approach the equilibrium set with a finite number of individual improvements are established, under conditions weaker than the better reply security, for three classes of strategic games: potential games, games with strategic complements, and aggregative games with appropriate monotonicity conditions.

1. Introduction

Reny (1999) made a significant step in the development of sufficient conditions for Nash equilibrium existence in games with discontinuous utility functions. A feature common to games considered by Reny and most of his followers, see, e.g., McLennan et al. (2011) or Prokopovych (2013), is that the strategy sets are convex and each utility function is quasiconcave in own argument. Bich (2009) relaxes the quasiconcavity, but not at all radically.

In this paper, we extend Reny's approach to three different classes of strategic games: potential games; games with strategic complements; aggregative games with appropriate monotonicity conditions. Besides, our attention is switched from the mere existence of a Nash equilibrium to the possibility to approach the equilibrium set with a finite "individual improvement path". What unites the three classes is that the existence of a Nash equilibrium in none of them has anything to do with convexity. Moreover, it is much easier to prove and understand in the case of a *finite* game; in an infinite game, there may be no equilibrium at all, to say nothing of its approachability, without some topological assumptions. And for each class of games, we obtain a set of such assumptions that could not be derived from the previous literature.

Following Reny (2016), we consider games with purely ordinal preferences, i.e., where utility functions take values from arbitrary chains rather than the real line. Inevitably, we only consider pure strategies. Our (i.e., essentially, Reny's) topological assumptions do not ensure the existence of the best responses; therefore, the standard fixed point theorems cannot be applied directly. Instead, we consider finite subgames, where Nash equilibria not only exist, but can be reached, starting from an arbitrary strategy profile, with a finite number of individual improvements. The "finite deviation" assumptions ensure the possibility to find a finite subgame every Nash equilibrium of which is arbitrarily close to the set of Nash equilibria of the original game. Thus, we obtain the "very weak finite improvement" property of the original game: the set of Nash equilibria is nonempty and can be approached with a finite number of individual improvements starting anywhere in the set of strategy profiles.

We understand potential games in a much broader sense than Monderer and Shapley (1996), viz. we consider games where individual improvements are acyclic. Thus, our Theorem 1 generalizes the main result of Kukushkin (2011), which in its turn generalized the good old "acyclicity plus open lower contour sets" theorem (Bergstrom, 1975; Walker, 1977). As an application to economics, we show that the assumptions of Theorem 1 hold in a rather general class of Bertrand competition games (Propositions 4.1 and 4.2).

Strategic complements are also understood in a more general, ordinal sense, as in Milgrom and Shannon (1994), rather than in the cardinal one, as in Vives (1990). Moreover, we do not fix a list of requirements a game must satisfy to deserve the badge of "Strategic Complements". The point is that there are various

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versions of the single crossing and quasisupermodularity conditions in the literature (Milgrom and Shannon, 1994; LiCalzi and Veinott, 1992; Shannon, 1995; Quah, 2007; Quah and Strulovici, 2009; Kukushkin, 2013b) and "trade-offs" between them are possible, i.e., a stronger interpretation of one property coupled with a weaker interpretation of the other may have the same implications as a weaker interpretation of the first property together with a stronger interpretation of the second. Our Theorems 2 and 2′ extend the main result of Kukushkin et al. (2005) to infinite games, even with some strengthening.

While the only known way to establish the existence of an equilibrium in a potential game of Section 4 or an aggregative game of Section 6 consists in following improvement paths, in the case of strategic complements there is also an option of invoking Tarski's fixed point theorem, which ensures equilibrium existence without giving much information on better, or even best, response dynamics (e.g., Theorem 5.1 of Vives (1990) establishes the convergence of Cournot tâtonnement to equilibrium only if the starting point belongs to a rather specific area in the set of strategy profiles). The fact that the mere existence of an equilibrium can be obtained under weaker assumptions than in our Theorem 2 may be of interest to some readers. (An anonymous referee even refused to see any value in studying improvement dynamics when the existence of an equilibrium can be established by other means.) Accordingly, a list of such assumptions is given in Propositions 5.1 and 5.2. A comparison with an earlier result on the existence of Nash equilibrium in a discontinuous game with a version of strategic complements, Theorem 2 of Prokopovych and Yannelis (2017), is in Section 7 (Remark 7.5).

In contrast to strategic complements, strategic substitutes, by themselves, are not conducive to the existence of Nash equilibrium. In a game with additive aggregation, however, they do ensure the existence of an equilibrium as was shown by Novshek (1985), see also Kukushkin (1994). Dubey et al. (2006), having modified a construction invented by Huang (2002) for different purposes, created a tool applicable to some non-additive aggregation rules as well. Kukushkin (2005) used the tool to show the convergence of Cournot tâtonnement to equilibrium in aggregative games exhibiting strategic complements, strategic substitutes, or a combination of both. The most general description of aggregation rules for which that trick can still work was given by Jensen (2010). Our Theorem 3 establishes the existence and approachability of Nash equilibrium in games with Jensen aggregation rules where the best responses may fail to exist.

Section 2 contains basic definitions and notations associated with a strategic game. In Section 3, we reproduce Reny's original notions and more general topological conditions, which, via a technical Proposition 3.4, play the key role in the rest of the paper. In Sections 4–6, we consecutively apply Proposition 3.4 to potential games, games with strategic complements, and aggregative games. Several related questions of secondary importance are discussed in Section 7. More complicated (or just tedious) proofs (of Propositions 3.2, 4.1 and 4.2, Theorems 2 and 3) are deferred to Appendix.

2. Basic definitions

A *strategic game* Γ is defined by a finite set of *players* N and, for each $i \in N$, a *strategy set* X_i, a chain $\mathcal{C}_i$ (a *utility scale*), and a "generalized" *utility function* $u_i : X_N \to \mathcal{C}_i$, where $X_N := \prod_{i \in N} X_i$ is the set of *strategy profiles*. For each $i \in N$, we denote $X_{-i} := \prod_{j \in N\setminus\{i\}} X_j$, and often use notation like $(x_i, x_{-i}) \in X_N$.

With every strategic game, we associate this *individual improvement relation* $\rhd^{\mathrm{Ind}}$ on X_N ($i \in N$, y_N, $x_N \in X_N$):

$$y_N \rhd_i^{\mathrm{Ind}} x_N \rightleftharpoons [y_{-i} = x_{-i} \,\&\, u_i(y_N) > u_i(x_N)];$$
$$y_N \rhd^{\mathrm{Ind}} x_N \rightleftharpoons \exists i \in N\, [y_N \rhd_i^{\mathrm{Ind}} x_N].$$

By definition, a Nash equilibrium is a *maximizer* of the relation $\rhd^{\mathrm{Ind}}$ on X_N, i.e., a strategy profile $x_N \in X_N$ such that $y_N \rhd^{\mathrm{Ind}} x_N$ holds for no $y_N \in X_N$. The set of Nash equilibria is denoted $\mathrm{E}(\Gamma) \subseteq X_N$.

An (*individual*) *improvement path* is a (finite or infinite) sequence $\langle x_N^k \rangle_{k=0,1,\ldots}$ such that $x_N^{k+1} \rhd^{\mathrm{Ind}} x_N^k$ whenever $k \geq 0$ and x_N^{k+1} is defined. A strategic game Γ has the *finite improvement property* (*FIP*, Monderer and Shapley (1996)) iff there is no infinite improvement path. Γ has the *weak finite improvement property* (*weak FIP*) iff, for every strategy profile $x_N^0 \in X_N$, there is a finite improvement path $x_N^0, \ldots, x_N^m$ such that $x_N^m \in \mathrm{E}(\Gamma)$. Obviously, FIP implies weak FIP: *every* improvement path in a game with FIP ends at a Nash equilibrium after a finite number of steps. Both properties look more natural for a finite game although they may be observed in an infinite game now and then.

Henceforth, the strategy sets X_i are assumed to be topological spaces; each chain $\mathcal{C}_i$ is endowed with its order interval topology; the sets X_N, $\mathcal{C}_N := \prod_{i \in N} \mathcal{C}_i$, X_{-i}, and $X_N \times \mathcal{C}_N$ are endowed with their product topologies. The topological closure of a subset Y of any one of those spaces is denoted $\mathrm{cl}Y$. We say that Γ has the *very weak FIP* (Kukushkin, 2011) iff, for every $x_N^0 \in X_N$, there is $y_N \in \mathrm{E}(\Gamma)$ such that for every open neighborhood O of y_N there is a finite improvement path $x_N^0, \ldots, x_N^m$ with $x_N^m \in O$. Slightly relaxing the requirement, we say that Γ has the *very–very weak FIP* iff, for every $x_N^0 \in X_N$, there is $y_N \in \mathrm{clE}(\Gamma)$ such that for every open neighborhood O of y_N there is a finite improvement path $x_N^0, \ldots, x_N^m$ with $x_N^m \in O$.

Remark. If X_N is a *metric space* with a metric d, then the very–very weak FIP can be reformulated as follows: for every $x_N^0 \in X_N$ and every $\varepsilon > 0$, there are $y_N \in \mathrm{E}(\Gamma)$ and a finite improvement path $x_N^0, \ldots, x_N^m$ such that $d(y_N, x_N^m) < \varepsilon$. In this case, the difference between the very weak FIP and the very–very weak FIP is whether the same $y_N \in \mathrm{E}(\Gamma)$ can be chosen for all $\varepsilon > 0$ or not.

Proposition 2.1. *A strategic game Γ has the very weak FIP if and only if, for every $x_N^0 \in X_N$ and every open neighborhood O of* $\mathrm{E}(\Gamma)$*, there is a finite improvement path $x_N^0, \ldots, x_N^m$ such that $x_N^m \in O$.*

Proof. The necessity is obvious: every open neighborhood of $\mathrm{E}(\Gamma)$ is simultaneously an open neighborhood of $y_N \in \mathrm{E}(\Gamma)$ from the definition of the very weak FIP. To prove the sufficiency, we suppose the contrary: for every $y_N \in \mathrm{E}(\Gamma)$, there is an open neighborhood $O(y_N) \ni y_N$ such that no finite improvement path started at x_N^0 ever reaches $O(y_N)$. Then we set $O := \bigcup_{y_N \in \mathrm{E}(\Gamma)} O(y_N)$; in the case of $\mathrm{E}(\Gamma) = \emptyset$, $O := \emptyset$. Now O is an open neighborhood of $\mathrm{E}(\Gamma)$; therefore, there must be a finite improvement path $x_N^0, \ldots, x_N^m$ such that $x_N^m \in O$. If $\mathrm{E}(\Gamma) = \emptyset$, we have $x_N^m \in \emptyset$; otherwise, there holds $x_N^m \in O(y_N)$ for some $y_N \in \mathrm{E}(\Gamma)$. In either case, we have a contradiction. □

Proposition 2.2. *A strategic game Γ has the very–very weak FIP if and only if, for every $x_N^0 \in X_N$ and every open neighborhood O of* $\mathrm{clE}(\Gamma)$*, there is a finite improvement path $x_N^0, \ldots, x_N^m$ such that $x_N^m \in O$.*

The proof is essentially the same as that of Proposition 2.1; only $\mathrm{E}(\Gamma)$ should be replaced with $\mathrm{clE}(\Gamma)$.

3. Better-reply security and finite deviation

We start with auxiliary notations. Considering functions u_i as components of a mapping $u_N : X_N \to \mathcal{C}_N$, we denote G the graph of the mapping, i.e., the set of pairs $\langle x_N, u_N(x_N)\rangle \in X_N \times \mathcal{C}_N$ for all $x_N \in X_N$. For every $x_N \in X_N$, we denote $\bar{G}(x_N) := \{v_N \in \mathcal{C}_N \mid (x_N, v_N) \in \mathrm{cl}G\}$ and perceive $\bar{G}$ as a correspondence from X_N to $\mathcal{C}_N$.

Then, we reproduce Reny's (1999) definitions. Player $i \in N$ *can secure a payoff of $\alpha \in \mathcal{C}_i$ at $x_N^* \in X_N$* iff there exists $y_i \in X_i$ such that