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Nash equilibrium uniqueness in nice games with isotone best replies

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HIGHLIGHTS

- Games with isotone best reply functions are considered.
- Chain-convexity and chain-concavity are introduced.
- Conditions that characterize (chain-)concave best reply functions are obtained.
- Nash equilibrium uniqueness results are provided.

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ABSTRACT

We prove the existence of a unique pure-strategy Nash equilibrium in nice games with isotone chain-concave best reply functions and compact strategy sets. We show a preliminary fixpoint uniqueness argument which provides sufficient assumptions on the best replies of a nice game for the existence of exactly one Nash equilibrium. Then we examine the necessity and sufficiency of the conditions on the utility functions for such assumptions to be satisfied; in particular, we find necessary and sufficient conditions for the isotonicity and concavity\chain-concavity of best reply functions. We extend the results on Nash equilibrium uniqueness to nice games with upper unbounded strategy sets and we present "dual" results for games with isotone convex\chain-convex best reply functions. A final extension to Bayesian games is exhibited.

1. Introduction

The issue of the existence of exactly one Nash equilibrium has been a point of interest since the inception of non-cooperative game theory. As is well known, any Nash equilibrium uniqueness result can obtain only for special classes of games. In this work, where such a type of result is investigated, we shall restrict attention to the class of *nice games*[1] with isotone best reply functions.

The "isotonicity" of best reply correspondences, in some loose sense, is a very general expression of the strategic complementarity among optimal choices of agents. Games with "isotone" best reply correspondences have received special attention in the economic and game-theoretic literature because of the richness and easy intelligibility of their equilibrium structure and properties. Such a literature, started from Topkis (1978, 1979), had been popularized in economics by several articles during the 1990s: Milgrom and Roberts (1990, 1996), Vives (1990) and Milgrom and Shannon (1994) just to mention a few. Some of these articles showed interesting properties implied by Nash equilibrium uniqueness in classes of games admitting isotone single-valued selections from best reply correspondences. For example, in some of such classes Nash equilibrium uniqueness was proved to be: equivalent to dominance solvability (see Theorem 5 and the second Corollary at p. 1266 in Milgrom and Roberts, 1990, Theorem 12 in Milgrom and Shannon, 1994 and Proposition 4 in Agliardi, 2000); sufficient to establish an equivalence between the convergence to Nash equilibrium of an arbitrary sequence of joint strategies and its consistency with adaptive learning processes (see the first Corollary at p. 1270 in Milgrom and Roberts, 1990 and Theorem 14 in Milgrom and Shannon, 1994); sufficient to infer the existence – and uniqueness – of coalition-proof Nash equilibria (see Theorem A1 and the last Remark at p. 127 in Milgrom and Roberts, 1996). The mentioned articles, however, do not provide sufficient structural conditions on the primitives of a game that ensure the existence of a unique Nash equilibrium.

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[1] I.e., games with a finite set of players whose strategy space is a closed proper real interval with a minimum and whose utility function is strictly pseudoconcave and upper semicontinuous in own strategy. The term *nice game* is introduced in Moulin (1984) and our definition is similar – but not identical – to the one therein.

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A recent strand of the literature on games on networks has investigated the issue of the existence of exactly one Nash equilibrium in some classes of games with isotone best reply functions; we mention in particular Belhaj et al. (2014) and Lagerås and Seim (2016). There is a fundamental difference in the manner in which sufficient conditions for Nash equilibrium uniqueness are provided by the two articles: in Lagerås and Seim (2016) the conditions are imposed on the utility function profile $(u_i)_{i\in N}$ of a nice game $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ with a set N of players while in Belhaj et al. (2014) directly on the joint best reply correspondence $(b_i)_{i\in N}$ of an *abstract game*[2] $\hat{\Gamma} = (N, (S_i)_{i\in N}, (b_i)_{i\in N})$ with a finite tuple $(S_i)_{i\in N}$ of compact real intervals as strategy sets. Evidently, a set of assumptions on the joint best reply correspondence $(b_i)_{i\in N}$ of an abstract game $\hat{\Gamma}$ that guarantee the existence of a unique fixpoint for that correspondence is in fact a set of assumptions that guarantee the existence of a unique Nash equilibrium for $\hat{\Gamma}$ and for every game Γ that generates that joint best reply correspondence $(b_i)_{i\in N}$. Unfortunately there does not exist a general method for converting a condition on $(b_i)_{i\in N}$ into a set of conditions on $(u_i)_{i\in N}$ and in many cases – like in that of concavity of best reply functions, which is imposed as an assumption in some results[3] by Belhaj et al. (2014) – the existing literature is of no help in indicating a method.

Like Belhaj et al. (2014), also this work establishes a Nash equilibrium uniqueness result for abstract games with isotone best reply functions that satisfy a notion of generalized concavity (unlike the just cited article, however, attention is not restricted to best reply functions that depend only on a weighted sum of the strategies). Established that result, we infer two corollaries (i.e., Corollaries 2 and 3 in Section 4) which will serve as a basis for our investigation and whose unified statement can be roughly rephrased[4] as follows.

> *Let $\hat{\Gamma} = (N, (S_i)_{i\in N}, (b_i)_{i\in N})$ be an abstract game in which the set N of players is finite, every strategy set S_i is a compact proper real interval and every best reply correspondence b_i is a function into S_i. Then $\hat{\Gamma}$ has exactly one Nash equilibrium if each best reply function b_i is (i) isotone, (ii) chain-concave (resp. chain-convex) and (iii) greater than* $\min S_i$ *(resp. smaller than* $\max S_i$*).*

The notion of chain-concavity\convexity invoked in the statement above – and defined in Section 3 – is new to the best of our knowledge and subsumes that of a concave\convex function being a proper generalization thereof: for the case of twice continuously differentiable functions a sufficient (but by no means necessary) easily verifiable condition that guarantees the chain-concavity\convexity is the nonpositivity\nonnegativity of the Hessian matrix (which is not sufficient to guarantee the concavity\convexity of a function of two or more variables). Nonetheless, but not in contrast with what just said, the general argument that underlies the previous Nash equilibrium uniqueness result is in some sense known and, according to Kennan (2001), should be probably ascribed to Krasnosel'skiĭ. Remarkably, such a result makes a crucial use of strategic complementarity: if the isotonicity condition of best reply functions were dropped without being replaced by some other condition then the modified "result" would be easily seen to be generally incorrect.

[2] Such a terminology is due to Vives (1990) according to Kukushkin (2016).

[3] In particular, the assumption of concavity of best reply functions appears in Proposition 1 in Belhaj et al. (2014). A stricter variant of concavity appears also in other papers (e.g., in Baetz, 2015 and – when best reply functions are not linear – in Hiller, 2012).

[4] Corollaries 2 and 3 are two fixpoint uniqueness results for a selfmap $f : \prod_{i\in I} F_i \to \prod_{i\in I} F_i$ but in fact, as we have already pointed out, they can be equivalently understood as two Nash equilibrium uniqueness results for the abstract game $(I, (F_i)_{i\in I}, (f_i)_{i\in I})$.

Armed with Corollaries 2 and 3, in Section 5 we show which conditions on the utility function u_i of a nice game with compact strategy sets are both necessary and sufficient for its associated joint best reply function b_i to satisfy the three conditions (i)–(iii) stated above: such a recovery – enunciated in Theorems 3 and 4 (and in their Corollaries 4 and 5) – is the main contribution of our work. The Nash equilibrium uniqueness results for nice games with compact strategy sets presented afterwards are implied as direct consequences, and the other equilibrium uniqueness results (for nice games with unbounded strategy sets and for Bayesian games in interim formulation) obtain by adaptation.

The paper is structured as follows: Section 2 recalls some preliminaries and clarifies some definitions; Section 3 exposits novel notions of generalized convexity\concavity; Section 4 shows a fixpoint uniqueness argument *à la* Krasnosel'skiĭ; Section 5 contains the main results on Nash equilibrium uniqueness of this article (i.e., Theorem 3 and Corollary 4) and highlights some characterizations of concave\convex and isotone\antitone best reply functions (i.e., Theorem 4 and Corollary 5); Section 6 relates our Nash equilibrium uniqueness results to known theorems of the literature, deals with the discontinuity of utility functions on the topological interior of the joint strategy set and shows an example of possible extension to games with incomplete information; Section 7 contains all proofs and auxiliary results.

2. Notation and preliminary definitions

Let A and B be nonempty subsets of two Euclidean spaces, $f : A \times B \to \mathbb{R}$ and $(a^*, b^*) \in A \times B$. Sometimes we write $f(\cdot, b^*)$ to denote the function $A \to \mathbb{R} : a \mapsto f(a, b^*)$ and $f(a^*, \cdot)$ to denote the function $B \to \mathbb{R} : b \mapsto f(a^*, b)$. Thus, for instance, the expression $f(\cdot, b^*)(a^*)$ is perfectly equivalent to the expression $f(a^*, b^*)$. Let I be a proper real interval and $g : I \to \mathbb{R}$. The upper (resp. lower) right Dini derivative of g at $x_0 \neq \sup I$ is denoted by $D^+g(x_0)$ (resp. $D_+g(x_0)$) and the upper (resp. lower) left Dini derivative of g at $x_0 \neq \inf I$ is denoted by $D^-g(x_0)$ (resp. $D_-g(x_0)$). Thus, for clarity, when ($A \subseteq \mathbb{R}$ and) we write $D^+f(\cdot, b^*)(a^*)$ – or an analogous expression – we mean to indicate the upper right Dini derivative of $f(\cdot, b^*)$ at a^*. We recall that Dini derivatives take values in the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. To avoid possible confusion, we clarify that in this work $\mathbb{R}_+$ and $\mathbb{R}_{++}$ respectively denote the nonnegative and positive real lines and that $\mathrm{int}(A)$ denotes the topological interior of A.

For real-valued functions, the following notions of monotonicity are in fact standard. In particular, for such a case, our definition of a quasiincreasing function coincides with the usual definition of a quasimonotone function (see, e.g., p. 1199 in Hadjisavvas and Schaible, 2009): we prefer to use the term quasiincreasing instead of quasimonotone in order to dually define quasidecreasing functions with a consistent terminology and to remark the fact that our definition is specialized to functions with totally ordered domains.

Definition 1. We say that $f : X \subseteq \mathbb{R} \to \overline{\mathbb{R}}$ is **increasing** (resp. **strictly increasing, decreasing, strictly decreasing**) iff $f(\underline{x}) \leq f(\overline{x})$ (resp. $f(\underline{x}) < f(\overline{x}), f(\underline{x}) \geq f(\overline{x}), f(\underline{x}) > f(\overline{x})$) whenever $\underline{x}, \overline{x} \in X$ and $\underline{x} < \overline{x}$.

Definition 2. We say that $f : X \subseteq \mathbb{R} \to \overline{\mathbb{R}}$ is **quasiincreasing** iff

$$\underline{x}, \overline{x} \in X, \underline{x} < \overline{x} \quad \text{and} \quad f(\underline{x}) > 0 \Rightarrow f(\overline{x}) \geq 0.$$

We say that $f : X \subseteq \mathbb{R} \to \overline{\mathbb{R}}$ is **quasidecreasing** iff $-f$ is quasiincreasing.

The map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x - 1$ if x is irrational and $f(x) = 0$ if x is rational is an instance of a quasiincreasing function. The map $f : \mathbb{R} \to \overline{\mathbb{R}}$ defined by $f(x) = -\infty$ if x is irrational and negative, $f(x) = +\infty$ if x is irrational and positive and $f(x) = 0$ if x is rational is a second instance.