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Stability Criterion for Nonsmooth Systems and Nash Equilibrium Seeking via Projected Gradient Dynamics ⋆

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Abstract: We derive a novel stability criterion for nonsmooth dynamical systems by virtue of a new set-valued Lie derivative of nonsmooth Lyapunov functions. This set-valued Lie derivative requires no computation of generalized gradients. Instead, it only calculates the directional derivatives. Moreover, our criterion allows for Lyapunov function candidates to be locally Lipschitz continuous and not necessarily regular. These merits strengthen the existing stability criterion and simplify its checking process. As an application, we establish the stability of projected gradient dynamics for distributed Nash equilibrium seeking under very mild conditions, compared to the existing ones.

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1. INTRODUCTION

Nonsmooth dynamics widely exist in nature, due to various factors such as mechanical friction and collision, hybrid or switching structure, and measurement constraints, referring to Cortés (2008); Shi and Hong (2009). Moreover, optimization and control in engineering design may yield nonsmooth systems, partially because they have appealing dynamics with preferable performance, with representative examples including sliding mode control, nonholonomic stabilization, and constrained resource allocation, see Utkin (1992); Yi et al. (2016).

Game theory receives much attention from the system and control community since it is a powerful tool for modeling and analyzing multi-agent cooperative control. Nash equilibrium seeking is one of the significant problems since in many engineering design, the Nash equilibrium is an acceptable solution. The so-called dynamic gradient play Shamma and Arslan (2005) is one of the most popular distributed protocols that render every player to reach the Nash equilibrium without the knowledge of the other players' cost functions. In general, such an equilibrium seeking dynamics can be treated as a nonsmooth projected gradient system.

Stability is one of the most important dynamic properties for natural or engineering systems, and Lyapunov-based methods play a key role in stability analysis. Smooth Lyapunov functions are suitable candidates for a class of nonlinear systems indicated by a series of converse theories Khalil (2002). However, sometimes, at most Lipshcitz continuous rather than C^1 Lyapunov functions can be guaranteed Rifford (2000). In order to deal with nonsmooth Lyapunov functions, nonsmooth analysis tools rather than the classical calculus should be employed. Perhaps the most well-known result is Bacciotti's stability criterion in terms of a so-called set-valued Lie derivative Bacciotti and Ceragioli (1999), which is based on the Clarke generalized gradients Clarke et al. (1998), and it is extended to allow nonpathological functions in Bacciotti and Ceragioli (2006). In addition, Guo and Huang (2009) proposes a robust stability criterion against perturbations for systems with discontinuous righthand side. Grzanek et al. (2008) gives a stability result via an integral approach without calculating Dini or Clarke generalized gradients. Nakamura et al. (2013) derives a nonsmooth feedback stabilization on general manifolds.

In spite of these developments, some further investigations are still needed to overcome existing shortcomings. For example, many well-developed gap or merit functions in optimization theory and variational analysis have the potential to serve as promising Lyapunov functions of dynamics for optimizations and equilibrium seeking, referring to Liang et al. (2016a,b, 2017). However, to calculate their generalized gradients is often impossible due to the complicatedness. Moreover, existing stability results require the Lyapunov function to have a regular property which is sometimes hard to check.

The objective of this paper is to propose a novel stability criterion for nonsmooth systems that overcome the difficulties mentioned above. The contributions of this paper can be summarized as follows:

- A new set-valued Lie derivative of nonsmooth Lyapunov functions is derived in terms of directional

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derivative rather than generalized gradients, which simplifies the stability checking process.

- A novel stability criterion is proposed by means of our set-valued Lie derivative. Moreover, no regular property of the Lyapunov functions is required in our criterion.
- Asymptotic stability of projected gradient dynamics for distributed Nash equilibrium seeking is obtained by our criterion under very mild conditions compared to existing results.

The rest of this paper is organized as follows. Section 2 gives some preliminaries on nonsmooth analysis and variational inequalities, while Section 3 proposes our novel stability criterion. Then Section 4 focuses on the distributed Nash equilibrium seeking problem and Section 5 provides two illustrative examples. Finally, Section 6 concludes the paper.

Notations: Let $\mathbb{R}$ and $\mathbb{R}_+$ be the sets of real numbers and positive real numbers, respectively. $\mathbb{R}^n$ stands for the n-dimensional real vector space. $B(x;\varepsilon) \subset \mathbb{R}^n$ is the open ball centered at x with radius $\varepsilon > 0$. $\|\cdot\|$ is the Euclidean norm. $\langle x, y\rangle$ denotes the inner product of two vector $x, y \in \mathbb{R}^n$ and $col\{x_1, x_2, \cdots, x_n\} = (x_1^T, x_2^T, \cdots, x_n^T)^T$ denotes the column vector stacked with column vectors x_i, $(i = 1, 2, \cdots, n)$.

2. PRELIMINARIES

In this section, we briefly introduce some nonsmooth analysis and variational inequalities, referring to Clarke et al. (1998) and Facchinei and Pang (2003).

2.1 Nonsmooth analysis

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be locally Lipschitz continuous if for any point $x \in \mathbb{R}^n$, there exist $K > 0$ and $\delta > 0$ such that

$$|f(y) - f(z)| \le K\|y - z\|, \quad \forall\, y, z \in B(x;\delta).$$

An important fact, known as Rademacher's Theorem, is that a locally Lipschitz function is (Frechet) differentiable almost everywhere.

Let f be a locally Lipschitz continuous function. The *right directional derivative* and *generalized directional derivative* of f at $x \in \mathbb{R}^n$ in the direction $v \in \mathbb{R}^n$ are

$$f'(x;v) \triangleq \lim_{h\to 0^+} \frac{f(x+hv) - f(x)}{h},$$

and

$$f^\circ(x;v) \triangleq \limsup_{y\to x,\, h\to 0^+} \frac{f(y+hv) - f(y)}{h},$$

respectively. The function f is said to be *regular* at point x provided that f is locally Lipschitz continuous at x and admits directional derivatives $f'(x;v)$ at x for all v, with $f'(x;v) = f^\circ(x;v)$. The *Clark generalized gradient* of f at x, denoted $\partial f(x)$, is defined as

$$\partial f(x) \triangleq \{\zeta \in \mathbb{R}^n \,|\, \langle \zeta, v\rangle \le f^\circ(x;v), \forall\, v \in \mathbb{R}^n\}.$$

Consider the following autonomous dynamical system

$$\dot{x}(t) \in F(x(t)), \quad x(0) = x_0, \tag{1}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued map and $\mathbf{0} \in F(x^*)$. A map $x(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$ is a solution to system (1) if it is absolutely continuous and satisfy (1) for $t > 0$ almost everywhere. Throughout this paper, we assume that there is at least one solution to (1). Sufficient conditions for the existence of solution can be found in (Aubin and Cellina, 1984, Chapter 2). System (1) is said to be *Lyapunov stable* at $x = x^*$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for each initial condition x_0, any solution $x(t)$ satisfies

$$\|x_0\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall\, t \ge 0.$$

Such a stability can be guaranteed if there exists a positive definite [1] and continuous Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ such that for all $t_1, t_2 \in \mathbb{R}_+$,

$$t_1 \le t_2 \Rightarrow V(x(t_2)) \le V(x(t_1)), \tag{2}$$

where $x(t)$ is any solution of (1). Moreover, if (2) holds for strict inequalities, then system (1) is asymptotically stable at $x = x^*$.

Condition (2) can be verified in terms of Lie derivative of V along the system equation (1) when $V(x)$ is a smooth function. However, in general, a stable system may lack of any smooth Lyapunov function. Then nonsmooth functions are often considered as Lyapunov function candidates. When a Lyapunov function is selected, the remaining question on stability checking is how to verify condition (2). A celebrated result, as a stability criterion, is stated in the following.

Lemma 1. (Bacciotti and Ceragioli (1999)). System (1) is Lyapunov stable at $x = x^*$ if there exists a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying the following conditions.

1. V is positive definite and regular;
2. $\forall\, x \in \mathbb{R}^n$, either $\max \mathcal{L}_F V(x) \le 0$ or $\mathcal{L}_F V(x) = \emptyset$, where $\mathcal{L}_F V$ is the set-valued Lie derivative of V with respect to F, defined as

$$\mathcal{L}_F V \triangleq \{a \in \mathbb{R} \,|\, \exists\, v \in F(x) \text{ such that } \langle p, v\rangle = a, \forall\, p \in \partial V(x)\}. \tag{3}$$

Remark 1. The stability criterion in Lemma 1 is very powerful provided that V is regular and its generalized gradient can be calculated. Also, the criterion remains valid for nonpathological functions [2], which include regular functions, semiconcave/semiconvex [3] functions and some function that is neither semiconcave nor semiconvex. However, in some situations, checking whether V is regular (or nonphysiological) and calculating its generalized gradient are not easy tasks. These observations motivate us to develop novel stability criterion that relaxes those checking tasks. We will present it in the next section.

2.2 Variational inequalities

Given a subset $\Omega \subseteq \mathbb{R}^n$ and a map $F : \Omega \to \mathbb{R}^n$, the *variational inequality*, denoted by $\mathrm{VI}(\Omega, F)$, is to find a

[1] Here, the positive definite means that $V(x^*) = 0$ and $V(x) > 0$ if $x \neq x^*$.

[2] A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be *nonpathological* if it is locally Lipschitz continuous and for any absolutely continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}^n$ and for almost every $t > 0$, the set $\partial V(\varphi(t))$ is a subset of an affine subspace orthogonal to $\dot{\varphi}(t)$, Bacciotti and Ceragioli (2006).

[3] A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be *semiconvex* if there exists a nondecreasing upper semicontinuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{\rho\to 0^+} \omega(\rho) = 0$ and $\lambda V(x) + (1-\lambda)V(y) - V(\lambda x + (1-\lambda)y) \ge -\lambda(1-\lambda)\|x - y\|\omega(\|x - y\|)$ for any pair $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$, Cannarsa and Sinestrari (2004).

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