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# Strong Nash equilibrium in games with common and complementary local utilities

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ABSTRACT

A rather general class of strategic games is described where the coalitional improvements are acyclic and hence strong Nash equilibria exist: The players derive their utilities from the use of certain facilities; all players using a facility extract the same amount of local utility therefrom, which amount depends both on the set of users and on their actions, and is decreasing in the set of users; the ultimate utility of each player is the minimum of the local utilities at all relevant facilities. Two important subclasses are "games with structured utilities," basic properties of which were discovered in 1970s and 1980s, and "bottleneck congestion games," which attracted researchers' attention quite recently. The former games are representative in the sense that every game from the whole class is isomorphic to one of them. The necessity of the minimum aggregation for the existence of strong Nash equilibria, actually, just Pareto optimal Nash equilibria, in all games of this type is established.



## 1. Introduction

Both motivation for and the structure of this paper closely resemble those of Kukushkin (2007). Moreover, the models considered in either paper, when described in very general terms, sound quite similarly.

The players derive their utilities from the use of certain objects. Rosenthal (1973) called them "factors"; following Monderer and Shapley (1996), we call them "facilities" here. The players are free to choose facilities within certain limits. All the players using a facility extract the same amount of "local utility" therefrom, which amount may depend both on the set of users and on their actions. The "ultimate" utility of each player is an aggregate of the local utilities obtained from all relevant facilities.

Four crucial differences should be listed at the start. First, in Kukushkin (2007), following Rosenthal (1973), each player summed up relevant local utilities (strictly speaking, monotone transformations were allowed); here, each player takes into account only the worst local utility (again, monotone transformations may be allowed).

Second, the main results of Kukushkin (2007) were about the acyclicity of *individual* improvements and, accordingly, the existence of Nash equilibria. Here, it is about the acyclicity of *coalitional* improvements and, accordingly, the existence of strong Nash equilibria.

Thirdly, we have to assume "negative impacts" here, i.e., whenever a new player starts to use a facility, those already there cannot be better off. In Kukushkin (2007), as well as in Rosenthal (1973), there was no need for such an assumption.

Finally, the games considered in Kukushkin (2007) were partitioned into two classes: "generalized congestion games" and "games with structured utilities". In the former class, the players choose *which* facilities to use and do not choose anything else; in the latter, each player chooses *how* to use facilities from a fixed list. Actually, the possibility of certain combinations was overlooked there, see Le Breton and Weber (2011), but the range of permissible combinations is rather limited in any case. Here, both those classes are present too, but "which" and "how" choices could be combined arbitrarily. It should be mentioned that, both here and in Kukushkin (2007), games with structured utilities form a representative subclass.

The idea of games with structured utilities and the minimum aggregation originated in Germeier and Vatel' (1974) although in a much less general form. Their approach was developed further in a series of papers, see Kukushkin et al. (1985) and references therein.

The first, to my knowledge, result on the existence of strong Nash equilibria in congestion games, even though without a reference to Rosenthal (1973), was in Moulin (1982, Chapter 5): pirates were going to a treasure island; each pirate could choose between two ships, and the more pirates on board of either ship, the slower

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it went. Since each player could only use a single facility (ship), the application of the minimum aggregation may be assumed, and hence that example belongs to the class of games considered here.

A systematic study of conditions under which a congestion game possesses strong Nash equilibria was started by Holzman and Law-Yone (1997), and has been continued (Holzman and Law-Yone, 2003; Rozenfeld and Tennenholtz, 2006; Epstein et al., 2009; Holzman and Monderer, 2015). As is natural in light of the necessity part of our Theorem 6.1, all those results need specific assumptions on available strategies.

The fact that the minimum ("bottleneck") aggregation and negative impacts in congestion games are conducive to coalition stability was gradually noticed quite recently (Fotakis et al., 2008; Harks et al., 2013). The results of those papers are rather similar to our Theorem 4.1, but obtained in a much less general models.

Here, the same fact is expressed in its most general form: As long as each player uses the minimum aggregation and there are negative impacts at each facility, it does not matter which subsets of facilities and what methods of using them are available to each player: all coalitional improvements are acyclic (to be more precise, there exists a "strong $\omega$-potential") and hence strong Nash equilibria exist and, in a sense, attract adaptive dynamics.

Theorem 4.4 shows that every game satisfying the assumptions of Theorem 4.1 is isomorphic to a game with structured utilities and the minimum aggregation. In other words, the main findings of Kukushkin et al. (1985) remain relevant to every model of this type that has been considered since then. That paper, however, was silent on some important issues, e.g., algorithmic and computational aspects.

Perhaps the most interesting results of this paper are Theorems 6.1 and 6.3, which establish the necessity of the minimum aggregation for the existence, regardless of other characteristics of the game, of Pareto optimal Nash equilibria, to say nothing of strong Nash equilibria, and hence for the acyclicity of coalitional improvements as well. The first result of this kind was in Kukushkin (1992); however, it was designed for a particular class of games, so rather peculiar combinations of the minimum and maximum were allowed, which are not good in a more general case.

The minimum operator is not at all unusual in the theory of production functions. Galbraith (1958, Chapter XVIII) explicitly invoked Leontief's model to justify an attitude to public and private consumption ("social balance") that sounds indistinguishable from the minimum aggregation. Our Theorem 4.1 shows that agents who have internalized this attitude do not need any taxes to provide for an efficient level of public consumption; it is difficult to say whether Galbraith himself expected such a conclusion.

Models of public good provision where the output of the public good is the minimum or maximum of private contributions ("weakest-link" or "best-shot") are considered now and then (Hirshleifer, 1983; Cornes and Hartley, 2007; Boncinelli and Pin, 2012). Such production functions have some nice implications in that context too, but not as good as here; in particular, the existence of a strong Nash equilibrium is not guaranteed.

Section 2 introduces principal improvement relations associated with a strategic game. Section 3 provides a formal description of our basic model as well as its main structural properties. Throughout Section 4, the players use the minimum aggregation. The main results there are Theorems 4.1 and 4.4.

In Section 5, we consider the maximum aggregation rule, which has the same implications in games with positive impacts (Theorem 5.1). The leximin/leximax aggregation of local utilities is also considered there. Its properties are much closer to those of additive aggregation than minimum/maximum ones; it ensures the acyclicity of individual improvements, but not of coalitional ones.

Section 6 contains the characterization results, Theorems 6.1 and 6.3, which establish the necessity of the minimum aggregation for the existence of Pareto optimal Nash equilibria under broad assumptions. In Section 7, several related questions of secondary importance are discussed.

More complicated proofs (of Theorems 2.1, 6.1 and 6.3) are deferred to Appendix.

## 2. Improvement dynamics in strategic games

A *strategic game* $\Gamma$ is defined by a finite set of players $N$ (we denote $n = |N|$), and strategy sets $X_i$ and utility functions $u_i$ on $X_N = \prod_{i \in N} X_i$ for all $i \in N$. We denote $\mathcal{N} = 2^N \setminus \{\emptyset\}$ (the set of potential coalitions) and $X_I = \prod_{i \in I} X_i$ for each $I \in \mathcal{N}$; instead of $X_{N\setminus\{i\}}$ and $X_{N\setminus I}$, we write $X_{-i}$ and $X_{-I}$, respectively. It is sometimes convenient to consider utility functions $u_i$ as components of a "joint" mapping $u_N : X_N \to \mathbb{R}^N$.

With every strategic game, a few improvement relations on $X_N$ are associated ($I \in \mathcal{N}, y_N, x_N \in X_N$):

$$y_N \triangleright_I x_N \iff \left[y_{-I} = x_{-I} \;\&\; \forall i \in I \left[u_i(y_N) > u_i(x_N)\right]\right]; \tag{1a}$$

$$y_N \triangleright^{\text{Ind}} x_N \iff \exists i \in N \left[y_N \triangleright_{\{i\}} x_N\right] \tag{1b}$$

(*individual improvement* relation);

$$y_N \triangleright^{\text{Coa}} x_N \iff \exists I \in \mathcal{N} \left[y_N \triangleright_I x_N\right] \tag{1c}$$

(*strong coalitional improvement* relation).

A *maximizer* of an improvement relation $\triangleright$, i.e., a strategy profile $x_N \in X_N$ such that $y_N \triangleright x_N$ holds for no $y_N \in X_N$, is an equilibrium: a Nash equilibrium if $\triangleright$ is $\triangleright^{\text{Ind}}$; a strong Nash equilibrium if $\triangleright$ is $\triangleright^{\text{Coa}}$.

An *individual improvement path* is a (finite or infinite) sequence $\{x_N^k\}_{k=0,1,\ldots}$ such that $x_N^{k+1} \triangleright^{\text{Ind}} x_N^k$ whenever $x_N^{k+1}$ is defined; an *individual improvement cycle* is an individual improvement path such that $x_N^m = x_N^0$ for $m > 0$. A strategic game has the *finite individual improvement property* (*FIP*; Monderer and Shapley, 1996) if there exists no infinite individual improvement path; then every individual improvement path, if continued whenever possible, reaches a Nash equilibrium in a finite number of steps.

Replacing $\triangleright^{\text{Ind}}$ with $\triangleright^{\text{Coa}}$, we obtain the definitions of a *coalitional improvement path*, a *coalitional improvement cycle*, and the *finite coalitional improvement property* (*FCP*). The latter implies that every coalitional improvement path reaches a strong Nash equilibrium in a finite number of steps.

**Remark.** Under our definitions, a single strategy profile is an improvement path (both individual and coalitional) by itself. This peculiarity causes no harm and is helpful in the formulation of Theorem 2.1.

For a finite game, the FIP (FCP) is equivalent to the acyclicity of the relation $\triangleright^{\text{Ind}}$ ($\triangleright^{\text{Coa}}$) and is equivalent to the existence of a "potential" in the following sense. An *order potential* of $\Gamma$ is an irreflexive and transitive relation $\succ$ on $X_N$ satisfying

$$\forall x_N, y_N \in X_N \left[y_N \triangleright^{\text{Ind}} x_N \Rightarrow y_N \succ x_N\right]. \tag{2}$$

A *strong order potential* of $\Gamma$ is an irreflexive and transitive relation $\succ$ on $X_N$ satisfying

$$\forall x_N, y_N \in X_N \left[y_N \triangleright^{\text{Coa}} x_N \Rightarrow y_N \succ x_N\right]. \tag{3}$$

In an infinite game, the absence of finite cycles does not mean very much by itself. One approach is to employ a more demanding notion of a potential. A binary relation $\succ$ on a metric space $X_N$ is *$\omega$-transitive* if it is transitive and the conditions $x_N^\omega = \lim_{k\to\infty} x_N^k$ and $x_N^{k+1} \succ x_N^k$ for all $k = 0, 1, \ldots$ always imply $x_N^\omega \succ x_N^0$.