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Equilibrium analysis of the all-pay contest with two nonidentical prizes: Complete results☆

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ABSTRACT

This paper studies contests in which three or more players compete for two nonidentical prizes. The players have distinct constant marginal costs of performance or bid, which are commonly known. We show that the contests have a generically unique Nash equilibrium, and it is in mixed strategies. Moreover, we characterize the equilibrium payoffs and strategies in closed form. We also study how the equilibrium payoffs and strategies vary with the prizes. As an application, we numerically compute the optimal allocation of prizes that maximizes the total expected bid of asymmetric players.

1. Introduction

Contests with asymmetric players and heterogeneous prizes are predominant. For example, students of various intellectual levels compete for different grades, athletes of different abilities compete for different medals, and employees with different experience compete for different promotion opportunities. If we rank the prizes in a contest from the highest value to the lowest, we obtain a nonincreasing sequence of prize values, to which we refer as the prize sequence. The prize sequences in these contests have different shapes. For instance, in the 2016 U.S. Open tennis tournament, the prize is \$3.5 million for the winner, \$1.75 million for a runner-up, and \$0.875 million for a semifinalist. A prize is roughly half of the value of the next higher prize. In contrast, the prizes in the golf tournaments do not have the same property. For example, in 2016 U.S. Open golf tournament, the prizes are \$1.8 million for the champion, \$1.1 million for the runner-up, and \$0.68 million for the third place.

The shape of the prize sequence is especially important if the players have different abilities. To see why, if the prize sequence is very concave, the difference between higher prizes is small relative to that between lower prizes, which leads to less competition among the players with stronger abilities. In contrast, if the prize sequence is very convex, the difference between lower prizes is small relative to that between higher prizes, which leads to less competition among the players with lower abilities.

In this paper, we consider a complete-information all-pay contest among players of distinct constant marginal costs and two prizes of distinct values. This is the simplest setup to introduce prize sequences of different concavity/convexity, measured in the ratio of the difference between the two prizes to the difference between the lower prize and zero. We show that the contest has a unique Nash equilibrium, and it is in mixed strategies. In addition, we provide a closed-form characterization of the equilibrium payoffs and strategies, and computer programs to numerically compute them.

This paper's contribution is threefold. First, it shows equilibrium uniqueness. The uniqueness is not obvious because multiple equilibria have been found in contests with identical players (e.g. Baye et al., 1996). In contrast, Siegel (2010) constructs a unique Nash equilibrium in contests with identical prizes and general nonlinear cost functions. This paper shows that his method, with non-trivial modifications, also applies to contests with asymmetric players and two distinct prizes, and can be used to show the uniqueness of Nash equilibrium.

Second, this paper provides a closed-form characterization of equilibrium payoffs and strategies in contests with two prizes of arbitrary values. As a result, it unifies the existing equilibrium characterizations with specific prize sequences, and we can illustrate how the unique equilibrium changes from one type to another as the prizes change. In addition, Xiao (2016) illustrates in an example that a convex prize sequence can lead to an equilibrium in which a player mixes over a non-interval set of bids. As a result of our closed-form characterization, we provide a necessary and sufficient condition for this to happen.

Third, this paper can be used to test conjectures on variants of all-pay auctions and contests as well as on their design questions. If

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there is significant heterogeneity among either players or prizes, it is typically difficult to characterize equilibria in these games. However, our closed-form characterization and computer programs can be used to test conjectures and determine what results to expect. Specifically, we numerically compute the optimal allocation of prizes that maximizes the total expected bid of three asymmetric players. We find that the resulting optimal prize sequence contains either a single prize or two equal prizes, which complements the existing results by examining all the marginal cost profiles in a simplex, including the extreme values of marginal costs that have been previously studied.

Literature There is a large literature on contests and, closely related, auctions. See Konrad (2009) for a comprehensive survey. This paper is closely related to auctions and contests with complete information. As in this paper, Nash equilibria in these setups are usually in mixed strategies. A variety of prize structures are studied. For example, there is a large literature on contests with a single prize (e.g., Baye et al., 1996; Che and Gale, 1998). Identical prizes are considered by Clark and Riis (1998) and Siegel (2009, 2010). Arithmetic prize sequences – with constant first-order differences – are studied by Bulow and Levin (2006) and González-Díaz and Siegel (2013).[1] Xiao (2016) considers geometric prize sequences, with a constant ratio of two consecutive prizes, and quadratic prize sequences, with constant second-order differences, where both sequences are convex.

The main difference of this paper from the above is that we consider both concave and convex prize sequences. Moreover, we consider how the concavity/convexity affects asymmetric players. Barut and Kovenock (1998) study arbitrary prize sequences in contests among identical players. This paper extends their setup to asymmetric players but restricts it to two distinct prizes. Our findings are different from theirs. We find a unique equilibrium in contrast to their multiple equilibria. In addition, the prize allocation affects the total expected bid in our setup while the total expected bid is independent of prize allocations in their setup.[2] Azmat and Möller (2009) also consider symmetric players in a study of competing contests. Sela (2012) studies sequential all-pay auctions with one object in each stage. Olszewski and Siegel (2016a) study heterogeneous prizes and asymmetric players in large contests where the numbers of prizes and players go to infinity. In contrast, this paper considers a similar contest but with a finite number of prizes.

There is a literature on contests with asymmetric information, in contrast to the complete information in this paper. For example, Rosen (1986) studies the role of convex prize sequences in single-elimination tournaments, in which the players' effort is not observable. Moldovanu and Sela (2001) study the optimal allocation of prizes for ex ante symmetric players. All-pay auctions between two ex ante asymmetric players are studied in various setups (e.g., Amann and Leininger, 1996; Lizzeri and Persico, 2000; Siegel, 2014; and Rentschler and Turocy, 2016). However, we cannot study the effects of different prize structures on asymmetric players in those setups because they have either a single prize or symmetric players. Parreiras and Rubinchik (2010, 2015) study all-pay auctions of multiple objects and multiple ex ante asymmetric players. In contrast to this paper, the equilibria in those auctions are in pure strategies.

The remainder of this paper is organized as follows. Section 2 introduces a contest model among three players. Section 3 characterizes the equilibrium payoffs, and Section 4 characterizes the equilibrium strategies. Section 5 generalizes the results to more than three players and studies the optimal prize allocation for asymmetric players.

2. Model

For simpler notation, Sections 2 to 4 focus on a contest with three players 1, 2, 3. Then, Section 5 extends the results to more players. Each player i has a constant marginal cost of bid $c_i > 0$, and the marginal costs are distinct $0 < c_1 < c_2 < c_3$.[3] Therefore, a bid $s_i \geq 0$ incurs a cost of $c_i s_i$ to player i. Player 1 is the strongest because it costs him the least to achieve the same bid. The contest has two distinct prizes $v_1 > v_2 > 0$.[4] Let $\mathbf{c} = (c_1, c_2, c_3)$ be the cost sequence and $\mathbf{v} = (v_1, v_2)$ be the prize sequence. Then, a contest is characterized by $(\mathbf{c}, \mathbf{v})$. The game is of complete information, so $(\mathbf{c}, \mathbf{v})$ is commonly known. Let the first order differences of the prizes be $\Delta_1 = v_1 - v_2$ and $\Delta_2 = v_2 - v_3$, where $v_3 = 0$. Then, the prize sequence is convex if $\Delta_1 > \Delta_2$, linear if $\Delta_1 = \Delta_2$, and concave if $\Delta_1 < \Delta_2$. We use the ratio Δ_1/Δ_2 to measure the convexity of the prize sequence, and we say a sequence is more convex than another if the ratio is larger.

Each player i chooses a bid $s_i \geq 0$ simultaneously. The player with the highest bid receives the highest prize v_1; the player with the second-highest bid receives the second-highest prize v_2; and the others receive no prize. In the case of a tie, ranks are allocated randomly with equal probabilities to tying players. For example, suppose $s_1 = s_2 > s_3$, then with probability 1/2, player 1 receives v_1 and player 2 receives v_2; and with probability 1/2, player 2 receives v_1 and player 1 receives v_2. If $s_1 > s_2 = s_3$, player 2 receives v_2 with probability 1/2, and player 3 receives v_2 with probability 1/2. If player i wins prize v_k with bid s_i, his payoff is $v_k - c_i s_i$; if a player chooses bid $s_i \geq 0$ but wins no prize, his payoff is $-c_i s_i$. All players are risk neutral. We consider only Nash equilibrium throughout the paper.

3. Equilibrium payoffs

We first introduce a sequence of definitions, and show in Proposition 1 that the equilibrium payoffs can be constructed using the definitions. After that, Proposition 2 characterizes equilibrium payoffs in closed form, and Corollary 1 discusses comparative statics of the equilibrium payoffs with respect to the prize sequence.

We use a c.d.f. $G_i : [0, +\infty) \to [0, 1]$ to represent player i's (mixed) strategy. The support of G_i is the smallest closed set to which G_i assigns probability 1. Before the discussion of equilibrium payoffs, we introduce some notation in a two-player contest and a three-player contest.

First, consider a two-player contest in which the top two players 1 and 2 compete for prizes v_1 and v_2. The two-player contest is isomorphic to a two-player complete-information all-pay auction, and it is well-understood.[5] The contest has a unique equilibrium, and it is in mixed strategies.[6] The equilibrium strategies are

$$G_1^2(s) = c_2 s/(v_1 - v_2) \tag{1}$$

$$G_2^2(s) = 1 - c_1/c_2 + c_1 s/(v_1 - v_2) \tag{2}$$

for $s \in [0, \bar{s}_1^2]$, where $\bar{s}_1^2 = (v_1 - v_2)/c_2$. Throughout the paper, superscripts indicate the number of players in a contest. Fig. 1 illustrates the equilibrium strategies if $c_1 = 1$, $c_2 = 4$ and $v_1 = 4$, $v_2 = 3$. The equilibrium payoffs are

$$u_i^2 = v_1 - (v_1 - v_2)c_i/c_2$$

for $i = 1, 2$.

[1] Bulow and Levin (2006) study labor markets in which firms compete for workers. Their model can be transformed into a contest with arithmetic prize sequences.

[2] More precisely, for a fixed budget of prize money, any prize sequence whose lowest prize is zero maximizes the total expected bid.

[3] If some players have identical marginal costs, there may be multiple Nash equilibria, so our uniqueness result does not apply. See, for instance, Baye et al. (1996).

[4] See Siegel (2010) for the case with $v_1 = v_2$.

[5] The contest is isomorphic to the complete-information all-pay auction with two players whose values are $(v_1 - v_2)/c_1$ and $(v_1 - v_2)/c_2$. The two games have the same equilibrium.

[6] See, for instance, Che and Gale (2006) and Kaplan and Wettstein (2006).