Journal of Mathematical Economics 74 (2018) 35–45

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Regularity of a general equilibrium in a model with infinite past and future[☆]

Alexander Gorokhovsky[a], Anna Rubinchik[b,*]

[a] Department of Mathematics, University of Colorado at Boulder, United States
[b] Department of Economics, University of Haifa, Israel

ARTICLE INFO

Article history:
Received 18 May 2017
Received in revised form 30 August 2017
Accepted 17 October 2017
Available online 10 November 2017

Keywords:
Overlapping generations
Implicit function theorem
Determinacy
Time-invariance
Comparative statics

ABSTRACT

We develop easy-to-verify conditions to assure that a comparative statics exercise in a dynamic general equilibrium model is feasible, i.e., the implicit function theorem is applicable. Consider an equilibrium equation, $\Upsilon(k, E) = k$ of a model where an equilibrium variable (k) is a continuous bounded function of time, real line, and the policy parameter (E) is a locally integrable function of time. The key conditions are time invariance of Υ and the requirement that the Fourier transform of the derivative of Υ with respect to k does not return unity. Further, in a general constant-returns-to-scale production and homogeneous life-time-utility overlapping generations model we show that the first condition is satisfied at a balanced growth equilibrium and the second condition is satisfied for "almost all" policies that give rise to such equilibria.

1. Introduction

Policy evaluation in a general equilibrium model with overlapping generations is perceived as a daunting task, and our objective is to provide a transparent way to make it more reliable. It is common, even in the case of a partial equilibrium analysis, to first look for equilibrium using numerical methods, then calibrate or estimate the parameters of the model to fit a given data set, and only then to perform policy "experiments", i.e., to evaluate the reaction of equilibrium variables given the calibrated parameters to a "shock" in policy, see, for example, Adjemian et al. (2011) for the description of the algorithm. In this paper we work with policies that are not limited to one-time shocks, rather, we allow them to be functions of time and of individual characteristics. We offer the ground work for an analytical approach: as in the classical textbook case à la Debreu, we show that implicit function theorem can be applied to an equilibrium equation ($\Upsilon(k, E) = k$) for "almost any" distribution of endowments (E) that yields a balanced growth equilibrium, in which case the equilibrium variables (e.g., k) can be viewed as smooth functions of endowments (E), thought of as a transfer policy.[1] Thus, the equilibrium responses generated by such comparative statics are data-independent and can be directly calculated from the specification of the model.[2]

The analytical approach we offer can be viewed as complimentary to the numerical one. Establishing local uniqueness (regularity) of equilibria validates the numerical approach by assuring that the close-by equilibrium after the policy change is unique; moreover, analytical approximation for the reaction of equilibrium to the policy change provides a robustness check for the parallel numerical calculation.

We find that the key sufficient condition for regularity of equilibria is *time-invariance*, implying, in particular, that time in the model should be the whole real line. Its alternative, the half-line, or assuming existence of some "starting point" at zero, prevents one from properly modelling perfect foresight, cf. Burke (1990), and might be responsible for indeterminacy, cf. Kehoe and Levine (1985) and Demichelis and Polemarchakis (2007), and hence inability to predict the effects of a policy. With time-invariance, verifying conditions needed for the implicit function theorem amounts to checking differentiability of the equilibrium map (Υ)

☆ The authors wish to thank for their comments L. Samuelson and the participants of the Seminar at the Department of Economics at the University of Haifa; Summer Workshop in Economic Theory in Paris, 2017; SAET 2017 and IFORS17. The scientific responsibility is assumed by the authors.

* Corresponding author.
E-mail addresses: Alexander.Gorokhovsky@colorado.edu (A. Gorokhovsky), arubinchik@econ.haifa.ac.il (A. Rubinchik).

1 See Mertens and Rubinchik (2017) for the overview of the existing applications of IFT to equilibria in infinite economies and the reasons why they cannot be applied to the overlapping generations models.

2 Note that objective here is distinct from a comparison across steady states or balanced growth paths, as in d'Albis (2007); Mierau and Turnovsky (2014): the latter can be useful, for example, in comparing across countries, each following its own path. Our analysis can be used to predict changes that will occur in the wake of an announcement of a new policy in a given economy.

https://doi.org/10.1016/j.jmateco.2017.10.006

and calculating Fourier transform of the derivative with respect to the endogenous variable (k).

This last step, based on Wiener's theorem, Wiener (1932), was also used in Mertens and Rubinchik (2017), where generic regularity and stability was demonstrated for a parametric version of an overlapping generations economy with Cobb–Douglas production. The genericity argument there was provided with respect to the parameters of the model.

The paper consists of two main parts. First we formulate the result for an arbitrary fixed point equation and then illustrate our approach using an overlapping generations model with a general smooth constant-returns-to-scale technology and assuming only time-separability and homogeneity of life-time consumption.[3] This rather minimal set of assumptions allows for balanced growth equilibria, as is well-known in the literature (King et al., 2002). This is a base-line equilibrium, to which the policy change is applied, which can then trigger a non-stationary equilibrium response. Characterization of the equilibria is based on Mertens and Rubinchik (2013). We demonstrate how to verify our sufficient conditions for the implicit function theorem in this case. In addition, we show that comparative statics is feasible for "almost any" transfer policy, i.e., the set of policies that is open and everywhere dense in the corresponding Banach space of policies. Proofs missing in the text, detailed presentation of the model (that summarizes Mertens and Rubinchik (2013)) and an explicit calculation of the equilibrium response of capital to a change in endowments are in the appendix.

2. The first formulation of the problem

Let P, V be the space of parameters and variables of an economic model respectively, both are Banach spaces. In the example that follows the space of parameters (transfers) is the space of locally integrable functions and the equilibrium variable (capital path) is a continuous uniformly bounded function defined on the real line (time).

Assume that for a given model, equilibrium conditions can be reduced to a fixed-point equation in variable $k \in V$, given parameters $E \in P$,

$$F(k, E) \stackrel{\text{def}}{=} \Upsilon(k, E) - k = 0 \tag{1}$$

The objective of the modeller is to assure that one can evaluate the equilibrium response of k to the change in the policy parameter E at some equilibrium k^0 given baseline policy E^0.

To illustrate the idea, let us view E as a function of two real variables: an individual characteristic (age) and time. The initial, base-line, equilibrium is stationary. There, E^0 is constant with time, though it can still be a non-trivial function of individual age, thus describing some fixed system of transfers across individuals of different ages, for example as in a pay-as-you-go pension system. Then the change in the policy parameter, δE, describes how the transfer system is altered across ages and over time. The resulting reaction of equilibrium variables, in particular capital path, is what we want to calculate.

For finite economies such calculations are done by appealing to an implicit function theorem (IFT), which requires differentiability of F and invertibility of the derivative $\frac{\partial F}{\partial k}$ at the base-line point (k^0, E^0). We follow the same route here. First, let us state the suitable version of the implicit function theorem (IFT). The differentiability below is in the sense of Fréchet.

[3] Such models are indispensable if one has to evaluate the effect of a policy that involves an intergenerational transfer, such as a pension reform for example, cf. e.g., de la Croix and Michel (2002) for the overview.

Theorem 1 (*Schwartz, 1997, Thm. 3.8.5.*). *Let $\mathscr{E}, \mathscr{F}, \mathscr{G}$ be three Banach spaces, F be a continuously differentiable map from an open set $O \subset \mathscr{E} \times \mathscr{F}$ into $\mathscr{G}$, $F : (k, E) \mapsto F(k, E)$. Let (k^0, E^0) be a point in O, $F(k^0, E^0) = 0$.*

If $\frac{\partial F}{\partial k}(k^0, E^0)$ is invertible in the space of linear operators from $\mathscr{E}$ to $\mathscr{G}$, then there exist open sets $A \subset \mathscr{E}$ and $B \subset \mathscr{F}$, $A \times B \subset O$ such that for every $E \in B$, there is a unique solution (in k) of the equation $F(k, E) = 0$ which belongs to A and there is a continuously differentiable function $\phi : B \to A$ such that $F(\phi(E), E) = 0$. Its derivative is given by

$$\phi'(E^0) = -\left(\frac{\partial F}{\partial k}(k^0, E^0)\right)^{-1} \circ \left(\frac{\partial F}{\partial E}(k^0, E^0)\right) \tag{2}$$

Remark 1. Notice that in the notation of Theorem 1, $\frac{\partial F}{\partial k}(k, E)$ is invertible in a neighbourhood of (k^0, E^0), since $(k, E) \mapsto \frac{\partial F}{\partial k}(k, E)$ is a continuous map invertible at $(k, E) = (k^0, E^0)$. It follows that there exists a neighbourhood N of E^0 such that $\frac{\partial F}{\partial k}(\phi(E), E)$ is invertible for any $E \in N$.

Now we have two tasks.

The first task is to find sufficient conditions for $\frac{\partial F}{\partial k}(k^0, E^0)$ to be invertible.

The second task is to show that invertibility holds for a *generic* set of parameters, i.e., for an open dense subset of P. The IFT already yields that the set is open given the differentiability of F, cf. Remark 1. The argument is completed in Section 4.3.

3. The first task

Our first task is to assure invertibility of the derivative of F, which defines the equilibrium condition (1), with respect to the endogenous variable (k) at the base-line equilibrium, k^0.

First, observe that if F is differentiable then its derivative can be represented as the following difference:

$$\frac{\partial F}{\partial k}(k^0, E^0) = \frac{\partial \Upsilon}{\partial k}(k^0, E^0) - I, \text{ where } I \text{ is the identity operator.} \tag{3}$$

Notice that the derivative $\frac{\partial \Upsilon}{\partial k}(k^0, E^0)$ is a bounded linear operator from V to V, where V is a Banach space. Invertibility of the difference as in (3) is closely related to the notion of a spectrum.

Definition 1. A spectrum of an operator T from a Banach space to itself is the set $\{z \in \mathbb{C} \mid T - zI \text{ is not invertible}\}$.

Hence, our task is reduced to verifying that unity ($z = 1$) is not in the spectrum of $\frac{\partial \Upsilon}{\partial k}(k^0, E^0)$, viewed as an operator. For that we use Wiener's theorem formulated for the space $L^\infty(\mathbb{R})$ of uniformly bounded functions on $\mathbb{R}$. The theorem implies that the spectrum of a convolution operator is a closure of the set of the values returned by its Fourier transform.

Definition 2.

(i) For $f \geq 0$ Lebesgue-measurable and a bounded measure μ the *convolution* $\mu \star f$ is the equivalence class of $t \mapsto \int \tilde{f}(t - s)\mu(ds)$ for any $\tilde{f} \in f$.

(ii) For a bounded measure μ, its *Fourier transform* (FT) is $\widehat{\mu}(\omega) = \int e^{i\omega t}\mu(dt)$ ($\widehat{g}$ for $g \in L_1$).

For a function $f \in L^1(\mathbb{R})$ define a bounded operator $B_f : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ by

$$B_f : g \mapsto f \star g$$

Theorem 2 (*Wiener, 1932*). *The spectrum of B_f is $\overline{\{\widehat{f}(\omega) \mid \omega \in \mathbb{R}\}} \cup \{0\} \subset \mathbb{C}$.*