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Rough approximations based on bisimulations

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ABSTRACT

In recent years, rough set theory initiated by Pawlak has been intensively investigated. When the classical rough sets based on equivalence relations have been extended to generalized rough sets based on binary relations, the lower and upper rough approximations, which are the core concepts of rough set theory, have been generalized in several different ways. A common feature of these generalized approximations is that they use only "one step" information of the underlying relation to discern objects. By "one step" in a binary relation we mean that the ordered pair of the starting and end points of the step belongs to the relation. Motivated by a rich notion, bisimulation, appearing in various areas of computer science, we introduce a kind of lower and upper rough approximations for generalized rough sets in this paper. Our lower and upper approximations are based on bisimulations, in particular, bisimilarity, which is the largest bisimulation. Roughly speaking, bisimilar objects are regarded as indiscernible. We present some basic properties of the new lower and upper rough approximations and illustrate our motivation and the applicability of our results by examples. Moreover, we make a detailed comparison between the rough approximations based on the underlying relation and the rough approximations based on bisimilarity. In particular, we provide a necessary and sufficient condition for the consistency of the two kinds of rough approximations.

1. Introduction

In order to cope with incomplete or inexact knowledge in information systems, Pawlak initiated the study of rough set theory in the early 1980s [33,34]. Since then, we have witnessed a world-wide growth of interest in the theory and its applications in many fields (see, for example, [35,36,52]). Nowadays, it becomes an excellent tool to handle granularity of data.

The starting point of rough set theory in [33,34] is that every object of the universe of discourse is associated with some information like data and knowledge, and the objects having the same information are indiscernible with respect to the available information. Formally, indiscernibility is an equivalence relation and any equivalence class is interpreted as a granule of knowledge. According to Pawlak's terminology, any subset X of the universe U is called a concept in U. If a concept X is a union of some equivalence classes, then X is precise; otherwise X is vague. The basic idea of rough set theory consists in approximating incomplete or inexact concepts with a pair of precise concepts—its lower and upper approximations.

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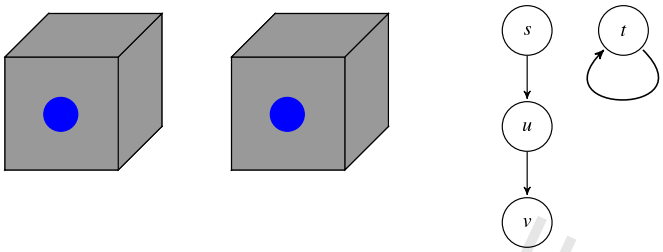

Fig. 1. Two black boxes and their behaviors. (For interpretation of the colors in this figure, the reader is referred to the web version of this article.)

However, as pointed out by many authors, an equivalence relation could fail to provide a realistic view of relationships among objects in real-world application. In other words, the requirement of equivalence relation as indiscernibility relation is too restrictive. In light of this, equivalence relation has been replaced by characteristic relation [14,15,38], similarity relation [43], tolerance relation [5,21,24,42], dominance relation and preorder [12,13], and even arbitrary binary relation [48, 49,51] in some extensions of the classical rough sets. At the same time, the corresponding lower and upper approximations have been intensively investigated. Besides the constructive approach extensively adopted in the literature mentioned above, some scholars (see, for example, [7,10,45,46]) considered algebraic approach in the study of rough sets based on binary relations.

In terms of rough sets based on general binary relations, there are a large number of lower and upper approximations (see, for example, [16, Section 2.3] and the references therein). Among others, Yao explored the lower and upper approximations that use successor or predecessor neighborhoods to replace equivalence classes in classical rough set theory [47, 49,50], in which some perfect constructive approaches and axiomatical characterizations have been provided. Taking a similarity relation into account, Slowinski and Vanderpooten [43] introduced a generalized definition of rough approximations based on the concept of ambiguity proposed by themselves. In [1], Abo-Tabl introduced three types of generalized rough approximations by using the intersection of right neighborhoods and compared these types with Yao's lower and upper approximations. In addition, Abu-Donia generalized the classical rough approximations by utilizing a family of different types of relations [2].

It is worth noting that in essence, the aforementioned generalized rough approximations based on binary relations are only dependent on "one step" information of the underlying relations. By "one step" in a binary relation we mean that the ordered pair of the starting and end points of the step belongs to the relation. Formally, for a relation $R \subseteq U \times U$ and any $s, t \in U$, we say that there is one step from s to t if $(s, t) \in R$. For instance, the elements that have one step from x give rise to the successor neighborhood $R_s(x) = \{y \in U \mid (x, y) \in R\}$ of x, and the elements that have one step to x form the predecessor neighborhood $R_p(x) = \{y \in U \mid (y, x) \in R\}$ of x. In general, the rough approximations via neighborhoods are based on the successor neighborhoods or predecessor neighborhoods. For example, the lower and upper approximations defined respectively by $\underline{app}X = \{x \in U \mid R_s(x) \subseteq X\}$ and $\overline{app}X = \{x \in U \mid R_s(x) \cap X \neq \emptyset\}$ are typical ones which are based on successor neighborhoods [47,49]. This implies that the indiscernibility is characterized by "one step" information of the underlying relation. However, in some rich and complex data sets, "one step" information may not be sufficient for discerning objects. To illustrate it, let us consider an example, which is inspired by [29, Chapter 2].

Example 1.1. Suppose that we want to discern reactive systems by interacting with them and observing the change of their states. Clearly, it is better to interact enough times instead of single time and then make a discrimination based on the observed sequence. Let us think of two reactive systems as two black boxes with one blue button each, as shown in Fig. 1. Following [29], we interact with the black boxes by trying to press the buttons. Sometimes the button goes down, which means that we succeed, and sometimes it does not. This is the only way that we can tell the difference between the two black boxes. For the first black box, we assume that its button can successively go down two times from its initial state and after that, it does not continue; for the second one, we assume that it can always go down. We describe their behaviors by the right state transition graph in Fig. 1. Formally, we are considering a generalized approximation space (U, R), where $U = \{s, u, v, t\}$ and $R = \{(s, u), (u, v), (t, t)\}$. Let us approximate the concept $X = \{s, t\}$ of their initial states. If we chose the above lower and upper approximation operators based on successor neighborhoods, we would obtain that $\underline{app}X = \{x \in U \mid R_s(x) \subseteq X\} = \{v, t\}$ and $\overline{app}X = \{x \in U \mid R_s(x) \cap X \neq \emptyset\} = \{t\}$. Recall that the basic idea of rough set theory consists in describing incomplete or inexact concepts with a pair of precise concepts which are granules of knowledge. Obviously, the states s, u, v, and t are different from one another; for instance, in state s the left button can successively go down two times, while in state u the left button can only go down one time. Therefore, each state should be regarded as a granule of knowledge. In light of this, approximating the concept $\{s, t\}$ by itself (namely, the union of two knowledge granules $\{s\}$ and $\{t\}$) is much better than using $\{v, t\}$ and $\{t\}$. However, the approximation operators in the literature cannot give rise to such an approximation. We will revisit it in detail by Example 3.1.

The above example motivates us to characterize indiscernibility by exploiting multi-step information. Fortunately, there is an elegant concept in computer science, called bisimulation [28,32,40], that reflects our basic idea. A bisimulation is a